Algebra 2

**Rational Expressions & Equations**

**Unit Summary:** In this unit, you will learn to add, subtract, multiply, and divide rational expressions. You will learn to solve equations with rational expressions to find the answer to the real-world problem.

**Lesson 2 – Rational Expressions**

**Key Words:**

* **Commutative Property** – a property of algebra that states that the order in which algebraic terms are added together does not affect the sum of those terms
* **Distributive Property** – a property of algebra that states that multiplying the sum of two or more addends by a number will give the same result as multiplying each addend individually by the number and then adding the products together; $a\left(b+c\right)=ab+ac$
* **domain** – the set of all possible inputs (x-values) of a function
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **quotient** – the result of a division problem
* **rational expression** – an expression that is the ratio of two polynomials
* **undefined** – an expression or value that has no meaning as a result of a zero in the denominator

**Formulas:**

* Commutative Property: $a+b=b+a $ and $ab=ba $
* Distributive Property:$a\left(b+c\right)=ab+ac$

**Objective 1:** In this section, you will identify rational expressions.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas**:

* A *rational expression* is an expression that is a fraction, or ratio , of two *polynomials*.



* Since it is a fraction, you will need to be sure the **denominator is not equal to zero**, otherwise the expression will be *undefined*.
* Recall that the *domain* is the set of all possible *x*-values of a function.
* An expression's *domain* will contain all values for the variable for which it is defined.
* To find the values that make the *rational expression* undefined, set the denominator equal to zero and solve for the variable.

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| **Example:** Which values cause the rational expression $\frac{x+2}{x^{2}-4x+4x}$ to be undefined? |
| **Step 1:** Set the denominator to zero.  | $$x^{2}-4x+4x=0$$ |
| **Step 2:** Solve for the variable. |  |
| **Step 3:** State the answer. | When $x=2 $, the rational expression will be undefined. The domain is all real numbers such that $x\ne 2 $.Written in set notation: $\left\{x:x ϵR,x\ne 2\right\}$ |

**Objective 2:** In this section, you will create equivalent rational expressions.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* Interpreting the fraction bar in a *rational expression* as division can help you write equivalent algebraic expressions.
* For example, the expression $\frac{6x^{4}}{3}$ can be interpreted as $6x^{4}÷3$.
	+ You can simplify the expression by finding a common factor between the numerator and denominator.
	+ 
* For more complex *rational expressions*, you can identify the common factors using polynomial identities and factoring.
* Common factors in the numerator and denominator divide to be 1, so they can be eliminated.

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| **Example:** Create an equivalent expression for the rational expression:$$\frac{4x^{2}-81}{8x-36}$$ |
| **Step 1:** Factor the numerator. | $4x^{2}-81$ follows the polynomial identity for a difference of square: $x^{2}-a^{2}=\left(x+a\right)\left(x-a\right)$.$$4x^{2}-81=\left(2x+9\right)\left(2x-9\right)$$Rewritten expression: $\frac{\left(2x+9\right)\left(2x-9\right)}{8x-36}$ |
| **Step 2:** Factor the denominator. | Both terms share 4, as a common factor. $$8x-36=4\left(2x-9\right)$$Rewritten expression: $\frac{\left(2x+9\right)\left(2x-9\right)}{4\left(2x-9\right)}$ |
| **Step 3:** Identify the common factor/s.  | $2x-9 $is in the numerator and denominator as a common factor. $$\frac{\left(2x+9\right)}{4}$$ |
| **Step 4:** Write the final equivalent expression. | $$\frac{4x^{2}-81}{8x-36}=\frac{2x+9}{4}$$ |
| **Step 5:** Determine which values of *x* will cause the expression to be undefined by setting the original denominator equal to zero and solving for *x*.  | *x* cannot equal $\frac{9}{2}$ because this will cause the expression to be undefined |

* Recall that, sometimes, long division is required to find the factors of a polynomial if the polynomial is not factorable.
* Sometimes, you are not able to divide the numerator and denominator evenly and the remainder is leftover.
	+ If there is a remainder, write the last term of the *quotient* with the remainder in the numerator and the divisor in the denominator (see example below).

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| **Example:** Create an equivalent expression for the rational expression:$\frac{3h^{2}-2h+1}{h-1}$  |
| **Step 1:** Determine if the numerator is factorable.  | The numerator does not appear to be factorable, so we must use long division.  |
| **Step 2:** Perform long division.  |  |
| **Step 3:** Write the equivalent expression.  | The remainder is 2, which you can write as a term in the quotient for a final answer of: |
| **Step 4:** Determine which values of *h* will cause the expression to be undefined by setting the original denominator equal to zero and solving for *h*. | $$h-1=0 $$$$h=1 $$*h* cannot equal 1 because this will cause the expression to be undefined |

**Objective 3:** In this section, you will compare rational expressions by writing them in different, equivalent forms.

*Mathematical Practice Standard: Look for and make use of structure.*

**Big Ideas:**

* Recall that using strategies like factoring, long division, and simplifying fractions might help you write an entire expression in its simplest form.
* If two expressions are written in their simplest form, then you can compare whether they are equivalent or not.
* Recall the following properties when rearranging terms to create equivalent expressions:
	+ *Commutative Property*: $a+b=b+a  $and $ab=ba $
	+ *Distributive Property*: $a\left(b+c\right)=ab+bc$

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| **Example:** Determine whether the following two rational expressions are equivalent:$\frac{x^{2}-6\left(x+12\right)}{-\left(12-x\right)}$ and $x+6 $ |
| **Step 1:** Begin with the first expression since it is more complex, and the second expression is as simplified as it can get.  | $$\frac{x^{2}-6\left(x+12\right)}{-\left(12-x\right)}$$ |
| **Step 2:** Apply the Distributive Property, factoring, and the Commutive Property. | Distributive Property:$$\frac{x^{2}-6\left(x+12\right)}{-\left(12-x\right)}=\frac{\left(x^{2}-6x-72\right)}{-12+x}$$Factoring:$$\frac{\left(x^{2}-6x-72\right)}{-12+x}=\frac{\left(x-12\right)\left(x+6\right)}{-12+x}$$Commutative Property:$$\frac{\left(x-12\right)\left(x+6\right)}{-12+x}=\frac{\left(x-12\right)\left(x+6\right)}{x-12}$$ |
| **Step 3:** Identify the common factors and simplify.  | $$\frac{\left(x-12\right)\left(x+6\right)}{x-12}=x+6$$ $\frac{x^{2}-6\left(x+12\right)}{-\left(12-x\right)}$ simplifies to $x+6 $. |
| **Step 4:** Determine which values of *x* will cause the expression to be undefined by setting the original denominator equal to zero and solving for *x*. | $$-12+x=0 $$$$x=12 $$When $x=12 $the expression will be undefined. |
| **Step 5:** State the answer. | The first expression $\frac{x^{2}-6\left(x+12\right)}{-\left(12-x\right)}$ simplifies to $x+6 $. Therefore, $\frac{x^{2}-6\left(x+12\right)}{-\left(12-x\right)}$ and $x+6 $are equivalent expressions except when $x=12 $. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | What value would cause the rational expression $\frac{8x^{2}+1}{x}$ to be undefined?*x* = \_\_\_\_\_ | 0 |
| P 2 | Which of the following options could be a denominator for a rational expression?Option #1: $x^{2}-x^{0.5}$Option #2: $8x\frac{1}{4}$Option #3: 5 | 5 |
| P 3 | Which of the following options is an equivalent form of the rational expression $\frac{6x^{3}+14x^{2}+8x}{2x}$?Option #1: $3x^{2}+7x+4$Option #2: $3x^{2}+7x^{2}+4x$Option #3: The rational expression has no equivalent forms. | 1 |
| P 4 | Which of the following is in its simplest form? Enter the option number of the correct answer.Option #1: $\frac{x}{x^{3}}$Option #2: $\frac{x+3}{x^{2}+9}$Option #3: $\frac{x^{2}-25}{x-5}$ | 2 |
| P 5 | Meghan simplified the expression $\frac{x^{2}-5x-24}{x^{2}-3x}$ incorrectly.First, she factored each of the terms: $\frac{(x-8)(x+3)}{x(x-3)}$.Next, she canceled the binomials in common: $\frac{(x-8)}{x}$Finally, she canceled any of the *x’s* that were in common and got the answer $\frac{(x-8)}{x}$.Which of the following options explains Meghan’s error?Option #1: Meghan factored incorrectly.Option #2: Meghan canceled the binomials incorrectly.Option #3: Meghan canceled the *x’s* incorrectly.Option #\_\_\_ explains Meghan’s error. | 2 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Gina is asked to identify a rational expression. Which of the following could she choose? | $$\frac{5.3}{2x-1}$$ |
| Q 2 | Which of the following is a rational expression whose domain has a restriction of $x\ne -1$? | $$\frac{x^{2}+6x+5}{6x+6}$$ |
| Q 3 | Marissa was asked to create an equivalent form of the rational expression $\frac{27h^{8}-18h^{5}+12h}{3h}$. Which of the following is a correct equivalent form? | $$9h^{7}-6h^{4}+4$$ |
| Q 4 | Melany was asked to create an equivalent form of the rational expression $\frac{-3x^{2}-15x-18}{x+3}$. Which of the following is a correct equivalent form? | $$-3x-6$$ |
| Q 5 | Express in simplest form: $\frac{36x-216}{x^{2}-36}$. | $$\frac{36}{x+6}$$ |

**Lesson 3 – Multiplication & Division of Rational Expressions**

**Key Words:**

* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **product** – the answer to a multiplication problem
* **quotient** – the result of a division problem
* **rational expression** – an expression that is the ratio of two polynomials

**Objective 1:** In this section, you will multiply rational expressions.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas**:

* Recall that rational expressions are just like fractions with a numerator and a denominator and there are strategies used to simplify or rewrite equivalent rational expressions:
	+ factoring out the greatest common factor
	+ factoring a trinomial
	+ factoring a difference of two squares
	+ dividing by a monomial
	+ polynomial long division
* When multiplying two polynomial expressions together follow these steps;
	+ 1. Factor each expression in the numerators and denominators, if possible and necessary.
	+ 2. Divide out all common factors.
	+ 3. Multiply the numerators. Multiply the denominators.

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| **Example:** Multiply the following rational expressions.$$\frac{12x^{3}-16x-4x}{18x-10x+\left(-4x\right)}⋅\frac{16x}{2x^{2}-8x}$$ |
| **Step 1:** Factor the numerator and denominator of the first expression. | The numerator is not factorable; however, each term does have a factor of *4x* in common.The denominator is all like terms, combine like terms: |
| **Step 2:** Factor the numerator and denominator of the second expression. | The numerator is already simplified. The denominator has a common factor of *2x*. |
| **Step 3:** Substitute the simplified expressions into the original expression. | $$\frac{4x\left(3x^{2}-4x-1\right)}{4x}⋅\frac{16x}{2x\left(x-4\right)}$$ |
| **Step 4:** Divide out common factors.  | $$\frac{\left(3x^{2}-4x-1\right)}{1}⋅\frac{8}{\left(x-4\right)}$$ |
| **Step 5:** Multiply the numerators and denominators. | $$\frac{\left(3x^{2}-4x-1\right)}{1}⋅\frac{8}{\left(x-4\right)}=\frac{8\left(3x^{2}-4x-1\right)}{x-4}$$$$\frac{8\left(3x^{2}-4x-1\right)}{x-4}=\frac{24x^{2}-32x-8}{x-4}$$ |
| **Step 6:** State the answer.  | The final product in simplified form is: $\frac{24x^{2}-32x-8}{x-4}$ |

**Objective 2:** In this section, you will divide rational expressions.

*Mathematical Practice Standard:  Make sense of problems and persevere in solving them.*

**Big Ideas:**

* When dividing two *rational expressions*, the *quotient* can be written as a multiplication problem, using the strategy “keep, flip, change”.
	+ For example: $\frac{2}{5}÷\frac{8}{3}=\frac{2}{5}⋅\frac{3}{8}$
	+ **Keep** the first expression – **Change** the operator to multiplication – **Flip** the fraction of the second expression.
* Once the division problem has been turned into a multiplication problem, follow the same process for multiplying polynomials.
	+ 1. Factor each expression in the numerators and denominators, if possible and necessary.
	+ 2. Divide out all common factors.
	+ 3. Multiply the numerators. Multiply the denominators.

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| **Example:** Divide the rational expressions. |
| **Step 1:** Rewrite the division problem as a multiplication problem using keep, flip, change. |  |
| **Step 2:** Factor each rational expressions numerator and denominator. | Three polynomials are factorable trinomials into binomials, and one is a binomial difference of squares factorable into two binomials. |
| **Step 3:** Divide out common factors. | Divide out common factors in the numerators and the denominators. |
| **Step 4:** Multiply the numerators together and the denominators together.  | Use the Distributive Property to multiply.$$\frac{\left(g+8\right)}{\left(g-4\right)}⋅\frac{\left(g-2\right)}{\left(g-3\right)}=\frac{g^{2}-2g+8g-16}{g^{2}-3g-4g+12}$$ |
| **Step 5:** Simplify. | Combine like terms.$$\frac{g^{2}+6g-16}{g^{2}-7g+12}$$ |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Mark multiplied the expression $\frac{3x^{2}-27}{6x+30}∙\frac{x^{2}+9x+20}{x^{2}-x-12}$ incorrectly.First, Mark factored each of the polynomial expressions: $\frac{3(x+3)(x-3)}{6x+5}∙\frac{(x+5)(x+4)}{(x+3)(x-4)}$.Next, Mark divided out common factors and got the answer: $\frac{x+3}{2}$.Determine Mark’s error.Option #1: Mark factored incorrectly.Option #2: Mark divided out common factors incorrectly.Option #3: Mark did not fully simplify the expression.Option # \_\_\_ shows Mark’s error. | 2 |
| P 2 | Given the rational expression $\frac{x^{2}+10x+16}{x^{2}-64}∙\frac{x^{2}+x-56}{2x+4}$, multiply and write the product in its simplest form.Option #1: $\frac{x-7}{2}$Option #2: $\frac{(x-7)(x+8)}{2(x-8)}$Option #3: $\frac{x+7}{2}$Option # \_\_\_ is correct. | 2 |
| P 3 | Given the rational expression $\frac{3x+18}{4x+8}∙\frac{x^{2}+2x}{x^{2}-36}$, multiply and write the product in its simplest form.Option #1: $\frac{1}{4(x-2)}$Option #2: $\frac{3x}{4(x+2)}$Option #3: $\frac{3x}{4(x-6)}$Option # \_\_\_ is correct. | 3 |
| P 4 | Marie divided the expression $\frac{x^{2}-4x-21}{6x^{2}}÷\frac{x^{2}-9}{x^{2}-3x}$ incorrectly.First, Marie took the reciprocal of the second term and multiplied: $\frac{x^{2}-4x-21}{6x^{2}}÷\frac{x^{2}-3x}{x^{2}-9}$.Next, Marie factored each of the terms: $\frac{(x-7)(x+3)}{6x^{2}}∙\frac{x(x-3)}{(x+3)(x-3)}$.Finally, Marie canceled terms that were in common and got the answer: $\frac{x(x-7)}{6x^{2}}$.Which of the following options explains Marie’s error?Option #1: Marie factored incorrectly.Option #2: Marie canceled incorrectly.Option #3: Marie did not fully simplify the expression.Option #\_\_\_ shows Marie’s error. | 3 |
| P 5 | Simplify the division problem: $\frac{2x-8}{x^{2}+x-12}÷\frac{20-5x}{x^{2}-9}$.Which of the following is correct?Option #1: $\frac{2(x+3)}{5(x+4)}$Option #2: $\frac{-2(x+3)}{5(x+4)}$Option #3: $\frac{-2(x-3)}{5(x+4)}$Option #\_\_\_ is the correct solution. | 2 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Multiply the rational expression and write the product in simplest form: $\frac{x^{2}+2x-3}{x^{2}+3x-4}∙\frac{x^{2}+6x+8}{x^{2}-9}$. | $$\frac{x+2}{x-3}$$ |
| Q 2 | Write the polynomial expression in simplest form: $\frac{x^{2}-9}{x^{2}-4}∙\frac{4x-8}{12x+36}$. | $$\frac{x-3}{3(x+2)}$$ |
| Q 3 | Express in simplest form: $\frac{x^{2}+8x-48}{x^{2}-144}÷(4-x)$ | $$\frac{-1}{x-12}$$ |
| Q 4 | Express in simplest form: $\frac{3x+18}{4x+8}÷\frac{x^{2}-36}{x+2}$. | $$\frac{2}{4x-24}$$ |
| Q 5 | Express in simplest form:$ \frac{4x^{2}-16x-84}{8x^{3}}÷\frac{x^{2}-9}{x^{2}-3x}$. | $$\frac{x-7}{2x^{2}}$$ |

**Lesson 4 – Addition & Subtraction of Rational Expressions**

**Key Words:**

* **Distributive Property** – a property of multiplication where each term of a sum inside a pair of parentheses gets multiplied by another term outside the parentheses
* **factoring** – the operation of resolving a quantity into factors
* **least common denominator (LCD)** – the smallest multiple that is common to all denominators of a set of fractions
* **least common multiple (LCM)** – the smallest quantity that is a multiple of two or more numbers or algebraic terms
* **like terms** – the terms in an expression that have the same variable to the same power
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **rational expression** – an expression that is the ratio of two polynomials

**Objective 1:** In this section, you will add rational expressions.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* Adding *rational expressions* is just like adding fractions.
* Follow these steps when adding *rational expressions*:
	+ 1. Identify or calculate a common denominator.
	+ 2. Write each rational term in an equivalent form with that common denominator.
	+ 3. Add the numerators together to form a single fraction and simplify.
* Recall how to find the common denominator of two expressions.
	+ Start by finding the *least common multiple (LCM)* of the denominators, also called the *least common denominator (LCD)*.
	+ List each unique factor the greatest number of times it appears in a single term, and multiply.

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| What is the least common multiple of $x,2x,2x^{2}$? |
| * List the prime factorizations of each term:
	+ $x=x $
	+ $2x=2⋅x $
	+ $2x^{2}=2⋅x⋅x$
* There are two unique factors: $x $and 2.
* $x $appears a maximum of **two** times in a single term, $x^{2}$.
* 2 appears a maximum of **once** in a single term, $2x $.
* LCM= $2⋅x⋅x=2x^{2}$
 |

* Recall that the LCD is the smallest multiple that is common to all denominators of a set of fractions.
* Adding Rational Expressions with a Common Denominator: If two rational expressions have a common denominator, you can simply add their numerators.

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| **Example:** Add both rational expressions. |
| **Step 1:** Factor the denominators of both expressions. | The denominator of the first expression is already factored. The denominator of the second expression can be rearranged using the Commutative Property and then factored. $$-81+x^{2}=x^{2}-81=\left(x+9\right)\left(x-9\right)$$Rewrite the problem: $$\frac{18-3x}{\left(x+9\right)\left(x-9\right)}+\frac{24+2x}{\left(x+9\right)\left(x-9\right)}$$ |
| **Step 3:** Add the numerators.  | The denominators are the same, so you only need to add the numerators. $$\frac{18-3x+24+2x}{\left(x+9\right)\left(x-9\right)}=\frac{42-x}{\left(x+9\right)\left(x-9\right)}$$ |
| **Step 4:** Simplify the denominator. | $$\frac{42-x}{\left(x+9\right)\left(x-9\right)}=\frac{42-x}{x^{2}-81}$$ |
| **Step 5:** State the answer. | The sum of the two rational expressions is $\frac{42-x}{x^{2}-81}$. |

* Adding Rational Expressions with Unlike Denominators: If two rational expressions do not have a common denominator, you can calculate the *least common denominator (LCD*).

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| **Example:** Add both rational expressions.$$\frac{\left(11-2z\right)}{z^{2}-6z+5}+\frac{z^{3}-3}{z-5}$$ |
| **Step 1:** Check to find out if the denominators are factorable. | The denominator of the first term is a factorable trinomial.$$z^{2}-6x+5=\left(z-5\right)\left(z-1\right)$$Rewrite the problem: $\frac{\left(11-2z\right)}{\left(z-5\right)\left(z-1\right)}+\frac{z^{3}-3}{z-5}$ |
| **Step 2:** Find the LCD. List each unique factor the greatest number of times it appears in a single term, and multiply. | * List each denominator:
	+ $\left(z-5\right)\left(z-1\right)$
	+ $\left(z-5\right)$
* There are two unique factors: $\left(z-5\right)$ and $\left(z-1\right)$
* $\left(z-5\right)$appears a maximum of one time in a single term.
* $\left(z-1\right)$ appears a maximum of one time in a single term.
* LCD=$\left(z-5\right)\left(z-1\right)$
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| **Step 3:** Make the denominators alike.  | Multiply the second term by $\frac{\left(z-1\right)}{\left(z-1\right)}$ so that the denominator matches the first term. |
| **Step 4:** Simplify the binomials in the numerator by using the Distributive Property. |  |
| **Step 5:** Combine into a single fraction and add the numerators.  |  |
| **Step 6:** Combine like terms. |  |

**Objective 2:** In this section, you will subtract rational expressions with like and unlike denominators.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* Subtracting *rational expressions* is just like subtracting fractions.
* Follow these steps when subtract *rational expressions*:
	+ 1. Identify or calculate a common denominator.
	+ 2. Write each rational term in an equivalent form with that common denominator.
	+ 3. Subtract the numerators to form a single fraction and simplify.
* Subtracting Rational Expressions with a Common Denominator: If two rational expressions have a common denominator, you can simply subtract their numerators.

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| **Example:** Subtract the rational expressions.$$\frac{3-6x}{5x-2}-\frac{3x-1}{5x-2}$$ |
| **Step 1:** Identify a common denominator.  | Both expressions have the same denominator $5x-2 $and are therefore common denominators. |
| **Step 2:** Subtract the numerators. Don’t forget to distribute the subtraction sign into the second term. | $$\frac{\left(3-6x\right)-\left(3x-1\right)}{5x-2}$$$$\frac{3-6x-3x+1}{5x-2}$$$$\frac{4-9x}{5x-2}$$ |
| **Step 3:** State the answer. | The result of subtracting the two rational expressions is$\frac{4-9x}{5x-2}$**.** |

* Subtracting Rational Expressions with Unlike Denominators: If two rational expressions do not have a common denominator, rewrite the expression using the least common denominator and then subtract the numerators.

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| **Example:** Subtract the rational expressions.$$\frac{1}{x^{2}-9}-\frac{2}{4\left(x+3\right)^{2}}$$ |
| **Step 1:** Factor the denominators. | Denominator 1: $x^{2}-9=\left(x-3\right)\left(x+3\right)$Denominator 2: $4\left(x+3\right)^{2}=4\left(x+3\right)\left(x+3\right)$Rewrite the expression: $$\frac{1}{\left(x-3\right)\left(x+3\right)}-\frac{2}{4\left(x+3\right)\left(x+3\right)}$$ |
| **Step 2:** Find the LCD. List each unique factor the greatest number of times it appears in a single term, and multiply. | * List each denominator:
	+ $\left(x-3\right)\left(x+3\right)$
	+ $4\left(x+3\right)\left(x+3\right)$
* There are three unique factors: $4, \left(x+3\right), and \left(x-3\right)$
* 4 appears a maximum of one time in a single term.
* $\left(x-3\right)$ appears a maximum of one time in a single term.
* $\left(x+3\right)$ appears a maximum of two times in a single term.
* The LCD is $4\left(x-3\right)\left(x+3\right)\left(x+3\right)=4\left(x-3\right)\left(x+3\right)^{2}$
 |
| **Step 3:** Make the denominators alike. | $$\frac{1}{\left(x-3\right)\left(x+3\right)}⋅\left(\frac{4\left(x+3\right)}{4\left(x+3\right)}\right) - \frac{2}{4\left(x+3\right)\left(x+3\right)}⋅\left(\frac{x-3}{x-3}\right)$$ |
| **Step 3:** Rewrite the expression using LCD. |  |
| **Step 4:** Subtract the numerators. Don’t forget to distribute the subtraction sign into the second term. |  |
| **Step 5:** Simplify |  |
| **Step 6:** State the answer. | The result of subtracting the two rational expressions is $\frac{x+9}{2\left(x-3\right)\left(x+3\right)^{2}}$. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Add the following rational expressions.$$\frac{2x+3}{x+2}+\frac{x-2}{x+2}$$ | $$\frac{2x+3}{x+2}+\frac{x-2}{x+2}=\frac{3x+1}{x+2}$$ |
| P 2 | Simplify the following rational expressions.$$\frac{5x}{x^{2}+5x+6}+\frac{x}{x+3}=\frac{?}{x^{2}+5x+6}$$ | $$\frac{5x}{x^{2}+5x+6}+\frac{x}{x+3}=\frac{x^{2}+7x}{x^{2}+5x+6}$$ |
| P 3 | Subtract the following rational expressions.$$\frac{\left(x+1\right)}{\left(x+3\right)}-\frac{5}{2x+6}=\frac{?}{2x+6}$$ | $$\frac{2x-3}{2x+6}$$ |
| P 4 | Find the difference of the following rational expressions.$$\frac{x^{2}-2}{x^{2}-5}-\frac{4x+1}{5-x^{2}}=\frac{?}{x^{2}-5}$$ | $$\frac{x^{2}+4x-1}{x^{2}-5}$$ |
| P 5 | Simplify the following rational expressions. Keep the denominator in its factored form.$$\frac{x^{2}}{(x-2)(x+5)}-\frac{x}{x-2}=\frac{?}{(?)(?)}$$ | $$\frac{-5x}{(x-2)(x+5)}$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Add the rational expressions to find the sum: $$\frac{x^{2}-2}{3x-2}+\frac{x+5}{3x-2}$$ | $$\frac{x^{2}+x+3}{3x-2}$$ |
| Q 2 | Add the rational expressions to find the sum: $\frac{-4}{x-4}+\frac{x^{2}-2x}{2x-8}$.Put the answer into its simplest form. | $$\frac{x+2}{2}$$ |
| Q 3 | Subtract the rational expressions to find the difference: $\frac{x^{2}-2}{4x-7}-\frac{x+4}{4x-7}$. | $$\frac{x^{2}-x-6}{4x-7}$$ |
| Q 4 | Subtract the rational expressions $\frac{x^{2}}{2x-12}-\frac{18}{x-6}$. Simplify the answer. | $$\frac{x+6}{2}$$ |
| Q 5 | What is the LCD in its factored form of the following rational expression subtraction?$$\frac{x^{2}+2x}{x^{2}+7x-8}-\frac{x}{x-1}$$ | $$(x-1)(x+8)$$ |

**Lesson 5 – Solving Rational Equations**

**Key Words:**

* **equivalent expressions** – expressions that have the same value or meaning, but do not look the same
* **extraneous solution** – a solution of a transformed equation that is not a solution of the original equation
* l**east common denominator (LCD)** – the smallest multiple that is common to all denominators of a set of fractions
* **least common multiple (LCM)** – the smallest quantity that is a multiple of two or more numbers or algebraic terms
* **like terms** – the terms in an expression that have the same variable to the same power
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **rate** – a ratio between two related quantities in different units
* **rational equation** – an equation containing at least one fraction whose numerator and denominator are polynomials
* **rational expression** – an expression that is the ratio of two polynomials

**Formulas:**

* Zero Product Property of Multiplication: If $ab=0 $, then $a=0, b=0 $, or both $a $and $b $equal to zero.
* Quadratic Formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$
* Multiplicative Property of Equality: If $a=b $then $a⋅c=b⋅c $

**Objective 1:** In this section, you will solve rational equations by creating equivalent expressions.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas**:

* Recall what you have learned about *rational expressions* to solve *rational equations* using equivalent expressions.
* The trick to solving rational equations is to find a common denominator for each side of the equation.
	+ This allows you to eliminate the denominators, set the numerators equal to one another, and solve the equation.
* A *rational equation* can have one or more *extraneous solutions*.
	+ An *extraneous solution* is a solution of a transformed equation not of the original one.
	+ You can check for *extraneous solutions* by plugging the answer to the equation back into the problem to determine if they make the equation true.

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| **Example:** Solve the following rational equation for *x*.$$\frac{x}{x+6}=\frac{72}{x^{2}-36}+4$$ |
| **Step 1:** Factor the denominator of the fraction on the right side of the equation. |  |
| **Step 2:** Set the factored denominator equal to zero and use the Zero Product Property of Multiplication to find the undefined values.  | Recall that the denominator can’t be zero. What values of *x* will make the fraction undefined? |
| **Step 3:** Find the LCD. List each unique factor the greatest number of times it appears in a single term, and multiply. | * List each denominator:
	+ $\left(x+6\right)$
	+ $\left(x+6\right)\left(x-6\right)$
* There are two unique factors: $\left(x+6\right) and \left(x-6\right)$
* $\left(x+6\right)$ appears a maximum of one time in a single term.
* $\left(x-6\right)$appears a maximum of one time in a single term.
* The LCD is $\left(x+6\right)\left(x-6\right)$
 |
| **Step 4:** Make the denominators match. |  |
| **Step 5:** Simplify and combine like terms. |  |
| **Step 6:** Set the numerators equal to each other. |  |
| **Step 7:** Solve for *x.* Recall that the quadratic formula can be used to solve for *x* in quadratic equations. |  |
| **Step 8:** Check for extraneous solutions for $x=-6 $. | Substitute –6 into the original equation to determine if the values make the original equation true. $x=-6 $is an extraneous solution since the result is division by zero, making it an undefined value. |
| **Step 9:** Check for extraneous solutions for $x=4 $. | Substitute 4 into the original equation to determine if the values make the original equation true.$x=4 $is a solution because the result is true. |
| **Step 10:** State the answer. | The only solution to the rational equation is $x=4 $. |

**Objective 2:** In this section, you will solve rational equations by multiplying both sides by a common denominator.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* Sometimes you must solve *rational equations* whose fractional terms contain variables in the denominator. Follow these steps when there are variables in the denominator:
	+ 1. identify undefined values
	+ 2. find the *least common denominator (LCD)*
	+ 3. multiply both sides of the equation by the *LCD*
		- Recall the Multiplicative Property of Equality that says you can multiply both sides of an equation by a non-zero quantity without changing the solutions to the equation.
	+ 4. simplify and solve.

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| **Example:** Solve the rational equation.$$\frac{3}{x-3}+\frac{2}{x+3}=\frac{10}{x^{2}-9}$$ |
| **Step 1:** Factor the denominator of the right side of the equation. | $$x^{2}-9=\left(x+3\right)\left(x-3\right)$$Rewrite the equation:$$\frac{3}{x-3}+\frac{2}{x+3}=\frac{10}{\left(x+3\right)\left(x-3\right)}$$ |
| **Step 2:** Identify any values that will make the equation undefined.  | Set each denominator to 0 and solve for *x.*$$x-3=0 $$$$x=3 $$$$x+3=0 $$$$x=-3 $$The undefined values are $x=3 $and $x=-3 $, therefore, $x\ne 3 $and $x\ne -3 $. |
| **Step 3:** Find the LCD. List each unique factor the greatest number of times it appears in a single term, and multiply. | * List each denominator:
	+ $\left(x-3\right)$
	+ $\left(x+3\right)$
	+ $\left(x+3\right)\left(x-3\right)$
* There are two unique factors: $\left(x+3\right) and \left(x-3\right)$
* $\left(x+3\right)$ appears a maximum of one time in a single term.
* $\left(x-3\right)$ appears a maximum of one time in a single term.
* The LCD is $\left(x+3\right)\left(x-3\right)$.
 |
| **Step 4:** Use the Multiplicative Property of Equality. Multiply each side of the equation by the LCD and simplify.  | $$3\left(x+3\right)+2\left(x-3\right)=10$$ |
| **Step 5:** Solve.  |  |
| **Step 6:** State the answer.  | The solution is not equal to 3 or –3, so the solution is $x=\frac{7}{5}$. |

**Objective 3:** In this section, you will use models involving rational equations to solve real-world problems.

*Mathematical Practice Standard: Attend to precision.*

**Big Ideas:**

* Recall that a *rate* compares two values in fractional form. Real-world problems involving rates usually require a rational equation to model.
* You can use a rational equation to solve any problem relating to an unknown rate, distance traveled, or time spent working on a job.
* When solving real-world problems, remember there are mathematical and real-world restrictions.
	+ Recall that in rational expressions, it is necessary to identify which values of the variable will cause the expression to be undefined (the denominator cannot be zero).

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| **Example:** Jeanette, Georgina, and Grace can build a tree house together in 10 hours. If Jeanette works alone, it will take her 28 hours. If Georgina works alone, she can complete the work in 24 hours. How many hours would it take Grae to build a tree house if she worked alone? |
| **Step 1: Write** each of the individual rates as rational expressions. |

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| Jeanette | $\frac{1}{28}$ tree houses/hour |
| Georgina | $\frac{1}{24}$ tree houses/hour |
| Grace | $\frac{1}{x}$ tree houses/hour |

 |
| **Step 2:** Model with a rational equation. | From the problem, we know that combined, they can complete a tree house in 10 hours, or $\frac{1}{10}$ tree houses per hour. |
| **Step 3:** Find the LCD. Find the LCD. List each unique factor the greatest number of times it appears in a single term, and multiply. | * List each denominator and factorization:
	+ $28=2⋅2⋅7 $
	+ $24=2⋅2⋅2⋅3 $
	+ $10=2⋅5 $
	+ $x=x $
* There are five unique factors: 2, 7, 3, 5, and *x*.
* 2 appears a maximum of **three** times in a single term.
* 7 appears a maximum of **one** time in a single term.
* 3 appears a maximum of **one** time in a single term.
* 5 appears a maximum of **one** time in a single term.
* $x $ appears a maximum of **one** time in a single term.
* The LCD is $2⋅2⋅2⋅7⋅3⋅5⋅x=840x $
 |
| **Step 4:** Use the Multiplicative Property of Equality. Multiply each side of the equation by the LCD and simplify. |  |
| **Step 5:** Solve for the variable. |  |
| **Step 6:** Interpret the answer. | Recall that x represented the amount of time Grace can build a tree house. The answer can be interpreted as: Grace can build a tree house on her own in about 44 hours.  |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Which of the following options is correct when you solve for all values of *x*:$$\frac{-9}{x+3}+\frac{3}{2}=\frac{3x}{6}-\frac{3}{2}$$Option #1: $x = 12$ and $x = -2$Option #2: $x = 12$Option #3: $x = -2$Option #4: no solution | 1 |
| P 2 | Identify the equation that shows $\frac{a}{a-5}-\frac{2}{a+5}=\frac{50}{a^{2}-25}$ correctly rewritten using its least common denominator. | 3 |
| P 3 | Solve the equation for *x*.$$\frac{10}{x−8}=\frac{2}{x−8}+16$$*x* = \_\_\_ | 8.5 |
| P 4 | You have 5 liters of Solution A, which is 50% diluted, and 15 liters of Solution B, which is 20% diluted. How much of Solution B must be added to Solution A to make Solution A 55% diluted? | The answer is extraneous. |
| P 5 | Greg is 1.7 times faster at cleaning windows than Owen. It takes 39 minutes for them to clean 100 windows together. Using a rational equation, calculate how long it would take Owen to clean 100 windows by himself. Round the answer to the nearest tenth.\_\_\_\_\_minutes | 105.3 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Solve for *x*: $\frac{4x}{x-3}=2+\frac{12}{x-3}$. | no solutions |
| Q 2 | Solve the equation for *x*.  $\frac{1}{x^{2}-x}+\frac{1}{x}=\frac{5}{x^{2}-x}$ *x = \_\_\_* | 5 |
| Q 3 | What is the least common denominator of the equation $\frac{5}{x+5}-\frac{1}{x^{2}+2x-15}=\frac{4}{x^{2}+2x-15}$? | $$(x+5)(x-3)$$ |
| Q 4 | Sumaya can manufacture a spark plug in 13 minutes. When her friend Leonardo helps her, it takes them 5.32 minutes. Using a rational equation, which of the following correctly calculates the rate Leonardo manufactures spark plugs, rounded to the nearest hundredth? | 9.01 minutes |
| Q 5 | Tabitha works at a rate of 1 unit every 10 hours. Working together with a coworker, it only takes 5.24 hours to make 1 unit. Which of the following models is a rational equation that could determine the rate at which the coworker produces units? | $$\frac{1}{10}+\frac{1}{x}=\frac{1}{5.24}$$ |