Algebra 2

**Polynomials**

**Unit Summary: In** this unit, you will learn about polynomials, how to operate on them, and how to use them to solve problems. Here are some examples of polynomials:

* $6x^{3}-18x^{3}+9x-1$
* $\frac{19p}{200}-p^{2}q^{7}g^{3}$

Most of the polynomials you will work with in this unit will contain one variable. You will add, subtract, multiply, and divide polynomials. You will generate equivalent polynomials, and factor polynomials. These are all the things you have done with numbers and expressions. Here are some mathematical strategies you learned previously:

* How can you identify and combine like terms within an expression?
* How can you factor out the greatest common factor from a list or sum of terms?
* What properties of algebra can you use to generate an equivalent mathematical expression?
* How does the algorithm of long division work?

The answers to these questions can also be applied to polynomials. By the end of this unit, you will have used polynomials to generate prime numbers and Pythagorean triples, written polynomials that describe patterns in a sequence of terms and predicted the coefficients of polynomial products.

**Lesson 2 – Successive Differences**

**Key Words:**

* **1st differences** – the set of numbers that are the differences between each term and its predecessor in a sequence
* **2nd differences** – the set of numbers that is the difference between each 1st difference and its predecessor
* **arithmetic sequence** – an ordered list of numbers in which the difference between any term and its predecessor is constant
* **common difference** – the difference between successive terms in an arithmetic sequence
* **degree** – the value of the highest exponent on any variable in a polynomial
* **factorial** – a function that multiplies a whole number by its preceding consecutive whole numbers
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **polynomial expression** – an expression with multiple terms of the form
* **polynomial sequence** – an ordered list of numbers in which each term is equal to the value of a given polynomial function at the corresponding term number
* **quadratic polynomial** – a polynomial, or expression that has the general form of $An^{2}+Bn+C$, where A, B, and C are real numbers
* **sequence** – a set of numbers that follow a specific pattern or formula
* **standard form** – an expression where the terms are written from highest power to the lowest power
* **successive differences** – the value being subtracted from consecutive terms in a sequence

**Formulas:**

* Standard Form of a Quadratic Polynomial: $$Ax^{2}+Bx+C$$

	+ Denoted as a sequence: $$A\_{n}=an^{2}+bn+c$$
* Standard Form of Cubic Polynomial: $$An^{3}+Bn^{2}+Cn+D$$
* First Difference: $$a\_{\left(n+1\right)}−a\_{n}$$
* Second Difference: $$\left[a\_{\left(n+2\right)}−a\_{\left(n+1\right)}−a\_{n}\right]$$
* Standard Form of a Polynomial: $$An^{k}+Bn^{k−1}+Cn^{k−2}+…$$
* Leading Coefficient: $$A=\frac{d}{k!}$$

**Objective 1:** In this section, you willshow that the 2nd differences of sequences from quadratic polynomials are constant.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* Recall the proper notations to use when working with terms and sequences.
	+ $$n= $$

	term, $$a\_{n}=$$

	term number
	+ for example, $$a\_{3}$$

	is the third term in the sequence, $$a\_{10}$$

	 is the tenth term in the sequence
* Recall that an *arithmetic sequence* is a list of numbers with a constant difference, called the *common difference*.
	+ To find the *common difference* in an *arithmetic sequence* we use the *1st difference*.
	+ The *1st difference* is the set of numbers that is the difference between each term and its predecessor in the sequence.
	+ 
	+ We use the equation $$a\_{\left(n+1\right)}−a\_{n}$$

	 to denote the common difference.
		- where $$a\_{n}$$

		 is a term in the sequence
		- and $$a\_{n+1}$$

		 is its next consecutive term
* *Quadratic polynomials* increase much more quickly than an *arithmetic sequence* and have both *1st differences* and *2nd differences*$a\_{n}$
	+ *2nd differences* are the set of numbers that is the difference between each *1st difference* and its predecessor.
	+ For any *quadratic polynomial sequence* of the form $$a\_{n}=an^{2}+bn+c$$

	 where $$a, b, and c $$

	are constants and $$a $$

	is not equal to 0, the *2nd differences* of the terms are $$2a $$

	.
* Follow these steps for calculating a *quadratic polynomial sequence:*

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| **Example:** Consider the quadratic polynomial sequence $$a\_{n}=n^{2}+3$$. Calculate for $$n=1,2,3,4,5. $$ |
| **Step 1:** Find the terms in the sequence by substituting the specified terms into the polynomial. If no terms are specified, use $$n=1,2,3,4,5.  $$ | Hint:* to find the first term of a sequence ($$a\_{1}$$), substitute 1 for *n* in the given expression.
* to find the second term of a sequence ($$a\_{2}$$), substitute 2 for *n* in the given expression.
* continue this process for all terms
 |
| **Step 2:** Find the 1st differences by subtracting each term from its previous term. |

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| In the previous step we found the following terms to be: | Calculate the 1st difference: |
| $$a\_{1}=4$$$$a\_{2}=7$$$$a\_{3}=12$$$$a\_{4}=19$$$$a\_{5}=28$$ | $$a\_{2}−a\_{1}=3$$$$a\_{3}−a\_{2}=5$$$$a\_{4}−a\_{3}=7$$$$a\_{5}−a\_{4}=9$$ |

Here is a visual to help:Notice that the 1st differences are not constant. So, we need to take the extra step of finding the 2nd difference. |
| **Step 3:** Find the 2nd difference by subtracting each *1st difference* from its predecessor. |

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| In the previous step we found the 1st differences: | Calculate the 2nd difference: |
|  | $$\left[a\_{3}−a\_{2}\right]−\left[a\_{2}−a\_{1}\right]=2$$$$\left[a\_{4}−a\_{3}\right]−\left[a\_{3}−a\_{2}\right]=2$$$$\left[a\_{5}−a\_{4}\right]−\left[a\_{4}−a\_{3}\right]=2$$ |

Here is a visual to help:Notice that the 2nd difference has a constant value of 2.  |

**Objective 2:** In this section, you will show that the *n*th differences of sequences from *n*-degree polynomials are a constant value.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* To identify the *degree* of a *polynomial*, you must first organize the *polynomial* in decreasing form using the or exponent value, of each term.
	+ The highest exponent value determines the *degree* of the *polynomial*.
* The *degree* of a *polynomial* shows how many differences must be performed until the differences are a constant value.

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| **Polynomial** | **Differences until a Constant Value** |
| 1st DegreeExample: $$2x+5 $$ | 1st Difference |
| 2nd DegreeExample: $$3x^{2}−2x+5$$ | 2nd Difference |
| 3rd DegreeExample: $$x^{3}+3x^{2}−2x+5$$ | 3rd Difference |
| $$n^{th}$$ degree | $$n^{th}$$ difference |

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| **Example:** Are the **3rd differences** a constant value for the following polynomial?$$a\_{n}=2n^{3}+n−1$$ |
| **Step 1**: Find the first terms in the sequence by substituting the *n*-values (use 1,2,3,4,5 when none are given) into the polynomial and evaluating.  |

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| ***n*** | $$a\_{n}=2n^{3}+n−1$$ | **Term Values** |
| 1 | $$a\_{1}=2\left(1\right)^{3}+\left(1\right)−1$$ | 2 |
| 2 | $$a\_{2}=2\left(2\right)^{3}+\left(2\right)−1$$ | 17 |
| 3 | $$a\_{3}=2\left(3\right)^{3}+\left(3\right)−1$$ | 56 |
| 4 | $$a\_{4}=2\left(4\right)^{3}+\left(4\right)−1$$ | 131 |
| 5 | $$a\_{5}=2\left(5\right)^{3}+\left(5\right)−1$$ | 254 |

 |
| **Step 2:** Use the term values to find the 1st differences by subtracting the current term from its previous term. |

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| **Term Value** | **1st Difference** |
| $$a\_{1}=2$$$$a\_{2}=17$$$$a\_{3}=56$$$$a\_{4}=131$$$$a\_{5}=254$$ | $$17−2=15 $$$$56−17=39 $$$$131−56=75 $$$$254−131=123 $$ |

 |
| **Step 3:** Find the 2nd difference by subtracting each1st difference from its predecessor. |

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| **1st Difference** | **2nd Difference** |
| $$a\_{2}−a\_{1}=15$$$$a\_{3}−a\_{2}=39$$$$a\_{4}−a\_{3}=75$$$$a\_{5}−a\_{4}=123$$ | $$39−15=24 $$$$75−39=36 $$$$123−75=48 $$ |

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| **Step 4:** Find the 3rd difference by subtracting each 2nd difference from its predecessor. |

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| **2nd Difference** | **3rd Difference** |
| $$\left[a\_{3}−a\_{2}\right]−\left[a\_{2}−a\_{1}\right]=24$$$$\left[a\_{4}−a\_{3}\right]−\left[a\_{3}−a\_{2}\right]=36$$$$\left[a\_{5}−a\_{4}\right]−\left[a\_{4}−a\_{3}\right]=48$$ | $$36−24=12 $$$$48−36=12 $$ |

The third difference is a constant value of 12. |

**Objective 3:** In this section, you will use successive differences to construct polynomial expressions from sequences.

*Mathematical Practice Standard: Use appropriate tools strategically.*

 **Big Ideas:**

* Once the *degree* of a polynomial has been established, you can write the *standard form of the polynomial*.
* By identifying the first few terms in function notation and the *successive differences*, you can construct a *polynomial expression* that represents the sequence.
* The *standard form of a polynomial* contains all terms that have exponents less than or equal to the *degree* of the *polynomial*.
	+ Standard Form: $$An^{k}+Bn^{k−1}+Cn^{k−2}+…$$
	+ Where *A, B, C*, etc. are constant values and *k* is the highest degree

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| Polynomial | Constant Differences | Standard Form |
| 1st Degree | 1st Difference | $$An+B $$ |
| 2nd Degree | 2nd Difference | $$An^{2}+Bn+C$$ |
| 3rd Degree | 3rd Difference | $$An^{3}+Bn^{2}+Cn+D$$ |
| 4th Degree | 4th Difference | $$An^{4}+Bn^{3}+Cn^{2}+Dn +E$$ |
| *kth* Degree | *kth* Difference | $$An^{k}+Bn^{k−1}+Cn^{k−2}+…$$ |

* The coefficient of the *highest degree term*, called the leading coefficient (*A*), can be determined from the *constant differences*.
	+ Formula: $$A=\frac{d}{k!}$$
	+ *k* is the degree of the polynomial
		- for example, for a third-degree polynomial, $$k=3 $$
	+ *d* is the constant difference
		- for example, for a third-degree polynomial, the value of the 3rd difference is *d*
* Putting it all together: You can construct the polynomial expression that represents the sequence when you have identified the following...
	+ 1. highest power
	+ 2. first few terms in function notation
	+ 3. leading coefficient

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| **Example:** Write a polynomial that represents the sequence.1, 12, 29, 52, ... |
| **Step 1:** Write the terms in function notation.  | $$f\left(1\right)=1$$$$f\left(2\right)=12$$$$f\left(3\right)=29$$$$f\left(4\right)=52$$ |
| **Step 2:** Find the highest power by identifying when the constant successive difference occurs.  |

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| **f(x)** | **1** | **12** | **29** | **52** |
| **1st differences** | $$12−1=11 $$ | $$29−12=17 $$ | $$52−29=23 $$ |  |
| **2nd differences** | $$17−11=6 $$ | $$23−17=6 $$ |  |  |

The constant successive differences occur in the 2nd difference with a constant value of 6. **This means the highest degree of the polynomial is 2.** |
| **Step 3:** Identify the form of the polynomial.  | Since the highest degree is 2, the form of the polynomial will be: $$An^{2}+Bn+C$$ |
| **Step 4:** Use function notation to identify the coefficients *A, B*, and *C*. Write and solve the system of equations. | Use the first three terms in the sequence to write three equations:To eliminate a variable, use the equation $$C=1−A−B $$:Find A by substituting $$B=11−3A $$:Now plug in $$A=3 $$to find the value for B:Pick any equation and plug in $$A=3 and B=2 $$to find the value for C: |
| **Step 5:** Write the expression for the polynomial using the values found for A, B, and C. | $$3x^{2}+2x−4$$ |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Find the 1st difference given the second and third term. Second term is $3a + 5$; third term is $8a - 3$. | $$5a -8$$ |
| P 2 | What are the 2nd differences of the sequence from the following polynomial?$$n^{2}+n+1$$ | 2 |
| P 3 | What should the degree of a polynomial sequence be so that its 6th differences are a constant value? | 6 |
| P 4 | *Use the image to answer the question.*A diagram shows four rows with numbers and blanks. Horizontal lines are shown between each number or blank in a row, and vertical lines extend from the horizontal lines to the numbers or blanks in the row below, indicating a relationship.The successive differences of a sequence are shown in this image. What is the degree of its associated polynomial expression? | 2 |
| P 5 | *Use the image to answer the question.*Picture 1, PictureWrite an expression in standard form that represents the sequence.\_\_\_\_\_ $x^{2}$ + \_\_\_\_\_*x* + \_\_\_\_\_ | 2, -3, 4 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which polynomial sequence shows that the terms of the 2nd differences are constant? | {3, 7, 13, 21} |
| Q 2 | Shayna, Jamal, and Anjali are finding the 2nd differences for the sequence with the formula $a\_{n}=n^{2}-3$. Shayna says the 2nd differences are a constant value of 5. Jamal says the 2nd differences are a constant value of 7. Anjali says the 2nd differences are a constant value of 2. Is Shayna, Jamal, or Anjali correct in finding the 2nd differences? | Anjali is correct. Jamal and Shayna both calculated 1st differences. |
| Q 3 | At which differences does the following polynomial sequence reach a constant value?$$a\_{n}=2n^{4}-n^{3}$$ | 4th differences |
| Q 4 | *Use the image to answer the question.*There are four rows of numbers with horizontal lines between the numbers and vertical lines extending to the numbers in the row below, indicating relationships.The successive differences of a sequence are shown in the image. What is the standard form of its related polynomial expression? | $$An^{3}+Bn^{2}+Cn+D$$ |
| Q 5 | *Use the image to answer the question.*There are five rows of numbers with horizontal lines between the numbers and vertical lines extending to the numbers in the row below, indicating relationships.Which expression in standard form represents the sequence? | $$2n^{4}+n^{2}+n+3$$ |

**Lesson 3 –** **Polynomial Multiplication**

**Key Words:**

* **binomial** – an algebraic expression with the sum or difference of two terms
* **Distributive Property** – a property of multiplication where each term of a sum inside a pair of parentheses gets multiplied by another term outside the parentheses
* **like terms** – terms in an expression that have the same variable to the same power
* **polynomials** – expressions with multiple terms
* **Product Rule of Exponents** – a rule stating that when multiplying same bases, add the exponents: $b^{m}\*b^{n}=b^{m+n}$

**Formulas:**

* Product Rule of Exponents: $$b^{m}⋅b^{n}=b^{m+n}$$
* Distributive Property: $$a\left(b+c\right)=ab+ac$$

**Objective 1:** In this section, you will multiply polynomials using a table.

*Mathematical Practice Standard: Reason abstractly and quantitatively.*

**Big Ideas**:

* Recall the *Distributive Property* used when multiplying two expressions together. This process can also be used when multiplying two *polynomials*.
* Recall the *Product Rule for Exponents* that states when you multiply the same bases, add the exponents.
	+ $$b^{m}⋅b^{n}=b^{m+n}$$
* Recall that *like terms* are terms in an expression that have the same variable to the same power.
* You may recall using the table or box method when multiplying *polynomials* together. The table method will work for *polynomials* of any degree.

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| **Example:** Multiply the expression $$\left(2x+3\right)\left(4x+5\right)$$ using a table. |
| **Step 1:** Create a table based on the expression given.  |

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|  | 2x | 3 |
| 4x |  |  |
| 5 |  |  |

 |
| **Step 2:** Evaluate each square by multiplying the term from that specific row and column together.  |  |
| **Step 3:** Add each product together to form a polynomial expression. | $$8x^{2}+12x+10x+15$$ |
| **Step 4:** Simplify the expression by combining *like terms*. | $$8x^{2}+\left(12x+10x\right)+15$$$$8x^{2}+22x+15$$ |

**Objective 2:** In this section, you will multiply polynomials using the distributive property.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas:**

* You have multiplied two polynomials using the box or table method, now you will use the *Distributive Property*. Both methods end in the same result.
* Recall the *Distributive Property* where each term of sum inside parentheses gets multiplied by another term outside of the parentheses.



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| **Example:** Find the product of the two polynomials $$\left(x+2\right)\left(x−3\right)$$. |
| **Step 1:** Distribute each term from the first polynomial onto the second.  |  |
| **Step 2:** Simplify the expression and combine like terms. |  |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Use the table to answer the question.A table is shown with 2 rows and 2 columns. The horizontal side is labeled 4 x cubed and 2 x squared while the vertical side is labeled 6 x and negative 9.Find the product of $\left(4x^{3}+2x^{2}\right)\left(6x-9\right)$. Provide your answer in descending order of exponents. | $$24x^{4}-24x^{3}-18x^{2}$$ |
| P 2 | Complete the table to find the product of $\left(5y^{2}-6y\right)\left(7y^{7}-y^{3}\right)$. Provide your answer in descending order of exponents. | 1. $35y^{9}$
2. $-42y^{8}$
3. $-5y^{5}$
4. $6y^{4}$
5. $= 35y^{9}-42y^{8}-5y^{5}+6y^{4}$
 |
| P 3 | Use the Distributive Property to multiply the following polynomials: $3x^{2}(2x^{4}-15x)$. | $$6x^{6}-45x^{3}$$ |
| P 4 | Find the product of $(3x^{2}-8)(4x^{2}+7)$. Provide your answer in descending order of exponents. | $$12x^{4}-11x^{2}-56$$ |
| P 5 | What is the product of the following polynomials: $(-3x^{5}-4x^{4})(7x^{2}-2x+6)$? | $$-21x^{7}-22x^{6}-10x^{5}-24x^{4}$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Use the table to answer the question.A table is shown with two rows and two columns. The horizontal side is labeled negative 4 x superscript 4 baseline and 10 x while the vertical side is labeled 5 x cubed and negative 1.Find the product when you multiply the polynomials $(-4x^{4}+10x)(5x^{3}-1)$. | $$-20x^{7}+54x^{4}-10x$$ |
| Q 2 | Construct a table to find the missing term in the product $\left(-9m^{6}+12m^{5}\right)\left(m^{2}+2m+1\right)= -9m^{8}-6m^{7}+?+12m^{5}$. | $$15m^{6}$$ |
| Q 3 | Multiply the polynomials $(6x^{4}+15s^{3})(6s^{3}-15s^{4})$ by constructing a table. | $$-90s^{8}-189s^{7}+90s^{6}$$ |
| Q 4 | Use the Distributive Property to multiply the polynomials $-5t^{3}(6t^{7}-9t)$. | $$-30t^{10}+45t^{4}$$ |
| Q 5 | Consider the product $(2x-x^{3})(-3x^{4}-7x^{2})$. Which of the following is the correct expression when each term of the first polynomial is distributed onto the second polynomial? | $$2x\left(-3x^{4}-7x^{2}\right)-x^{3}(-3x^{4}-7x^{2})$$ |

**Lesson 4 –** **Polynomial Division**

**Key Words:**

* **dividend** – the top part of a rational expression; the value that is being divided
* **divisor** – the bottom part of a rational expression; the value that the dividend is being divided by
* **domain** – the set on which a function is defined
* **inverse** – an operation that undoes the effect of another operation
* **long division** – a process by which quantities are divided by taking the difference until there is a remainder

**Objective 1:** In this section, you will divide polynomials by recognizing division as the inverse operation of multiplication.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas**:

* Recall that multiplication and division are *inverses* of each other.
* Just like using numbers to show inverses, you can also use polynomials.
* Dividing a polynomial by a certain number is the same as multiplying the polynomial by the *inverse* of that number.
	+ For example:

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| $$\frac{6x^{2}−3x−9}{3}$$ | is equivalent to | $$\frac{1}{3}\left(6x^{2}−3x−9\right)$$ |

* When polynomials are multiplied together, the two polynomials multiplied are factors of their product.

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| Simplify the expression.$$\frac{x^{2}−3x−10}{x−5}$$* The term on the bottom cannot easily divide into the expression on top. Factor the top to see what factors make up the expression.

$$\frac{\left(x+2\right)\left(x−5\right)}{x−5}$$* Therefore: $$\frac{x^{2}−3x−10}{x+2}=x−5$$ and $$\frac{x^{2}−3x−10}{x−5}=x+2$$
 |

**Objective 2:** In this section, you will divide polynomials using long division in a way analogous to dividing numbers.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas:**

* Just like dividing whole numbers using common factors, you can divide polynomials using common factors.
* To perform polynomial *long division*, follow these steps:
	+ Example: 

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| **Steps** | **Example** |
| **Step 1**: Set up the two polynomials to be divided as shown, with the dividend inside and the divisor outside. |  |
| **Step 2:** Divide the first two terms of the expressions with a focus on the leading terms.How many times does the leading term of the divisor divide into the leading term of the dividend?Place the resulting term above the dividend.  | How many times does $$x $$go into $$x^{2}$$? |
| **Step 3:** Multiply the result by the divisor and place the product under the dividend.  | $$x⋅\left(x+3\right)=x^{2}+3x$$ |
| **Step 4:** Subtract the resulting expression from the dividend.  | $$\left(x^{2}+2x−3\right)−\left(x^{2}+3x\right)=−x−3$$ |
| **Step 5:** Divide the first term of the remaining expression by the first term in the divisor. Place the resulting term above the dividend. | How many times does $$x $$go into $$−x $$? |
| **Step 6:** Multiply the result by the divisor and place the product under the dividend. | $$−1⋅\left(x+3\right)=−x−3$$ |
| **Step 7:** Subtract the resulting expression from the remaining dividend. | $$\left(−x−3\right)−\left(−x−3\right)=0$$ |
| **Step 8:** State the answer. Since the final expression is 0, or canceled out, the two expressions can be divided evenly with the result above the dividend.  |  |

* If the final difference in *long division* is not zero, you can infer that that the *divisor* is not a factor of the *dividend*.
	+ For example, in the following problem, you can infer that $$x+3 $$

	is not a factor of $$x^{2}−6x+8$$

	 because the resulting difference is not zero.
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**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Simplify $\frac{4x^{2}-12x+24}{4x}$. Write your answer in standard form. | $$x-3+\frac{6}{x}$$ |
| P 2 | Simplify $\frac{x^{2}+5x+6}{x+2}$. Write your answer in standard form. | $$x + 3$$ |
| P 3 | Use long division to divide the polynomial $24x^{4}-24x^{3}-18x^{2}$ by $4x^{3}+2x^{2}$. Write your answer in standard form. | $$6x - 9$$ |
| P 4 | Use long division to divide the polynomial $33x^{5}+22x^{4}-50x^{3}-26x^{2}+13x$ by $3x^{2}+2x-1$. Write your answer in standard form. | $$11x^{3}-13x$$ |
| P 5 | Use long division to divide the polynomial $-30x^{6}+48x^{5}+50x-80$ by $5x-8$. Write your answer in standard form. | $$-6x^{5}+10$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which of the following worked equations demonstrates that you can divide polynomials by recognizing division as the inverse operation of multiplication? | $$\frac{8x^{2}-4x+12}{4x}=(\frac{1}{4x})(8x^{2}-4x+12)$$ |
| Q 2 | Which of the following is the simplified form of $\frac{18x^{2}-12x+6}{3x}$? | $$6x-4+\frac{2}{x}$$ |
| Q 3 | Use long division to divide the polynomial $15x^{2}+14x-8$ by $3x+4$ . What is the quotient? | $$5x-2$$ |
| Q 4 | Which of the following expressions is a factor of $2x^{2}+9x-35$. Use long division to solve. | $$2x - 5$$ |
| Q 5 | Which of the following is not a factor of $3x^{3}-10x^{2}-143x-90$? Use long division to solve. | $$x - 6$$ |

**Lesson 5 – Polynomial Operations**

**Key Words:**

* **Commutative Property** – a property of algebra that states that the order in which algebraic terms are added together does not affect the sum of those terms
* **degree** – the value of the highest exponent on any variable in a polynomial
* **order of operations** – the order in which arithmetic operations are performed in an expression
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **Product Rule of Exponents** – a rule stating that when multiplying same bases, add the exponents: $b^{m}\*b^{n}=b^{m+n}$
* **Quotient Rule of Exponents** – a rule stating that when dividing same bases, subtract the exponents: $b^{m}/b^{n}=b^{m-n}$

**Formulas:**

* Commutative Property:
	+ Addition: $$a+b=b+a $$
	+ Multiplication: $$ab=ba $$
* Product Rule of Exponents: $$b^{m}⋅b^{n}=b^{m+n}$$
* Quotient Rule of Exponents: $$\frac{b^{m}}{b^{n}}=b^{m−n}$$

**Objective 1:** In this section, you will perform addition, subtraction, multiplication, and division on polynomials.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas**:

* Addition, subtraction, multiplication, and division are operations that can be used on polynomial expressions. These operations will often be combined when simplifying a polynomial expression.
* By following the *order of operations* and *combining like terms* to simplify, it is possible to find the sum and difference of *polynomial* expressions.
	+ Recall the order of operations: **P**arentheses $$\rightarrow  $$

	**E**xponents$$\rightarrow  $$

	**M**ultiplication$$\rightarrow  $$

	**D**ivision$$\rightarrow  $$

	**A**ddition$$\rightarrow  $$

	**S**ubtraction
* Recall the steps and rules for [multiplying](#Bookmark1) and [dividing](#Bookmark2) polynomials.
* Recall the following exponent rules:
	+ Product Rule of Exponents: $$b^{m}⋅b^{n}=b^{m+n}$$
	+ Quotient Rule of Exponents: $$\frac{b^{m}}{b^{n}}=b^{m−n}$$
* Recall the *Commutative Property* for addition and multiplication.
	+ The property states that the order in which terms are added or multiplied together does not affect the sum.
	+ $$a+b=b+a $$
	+ $$ab=ba $$

**Objective 2:** In this section, you will determine the first and last terms of polynomials resulting from addition, subtraction, multiplication, and division without performing the operations completely.

*Mathematical Practice Standard: Reason abstractly and quantitatively.*

**Big Ideas:**

* Sometimes you will be asked to describe the behavior of a *polynomial*. This can be done by considering the first and last terms.
	+ The first term is the highest *degree* of the expression.
	+ The last term is the lowest *degree* of the expression.



* To determine the first term of a solution from a *polynomial* operation is to identify the leading terms of each *polynomial* and combine using the rules of operations.
	+ Recall that when adding and subtracting polynomials, the exponent does not change.
	+ Recall that when multiplying or dividing polynomials, the exponent may change, according to the [product and quotient rules of exponents](#Bookmark3).

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| **Example:** Determine the first term of the solution for the following operation. |
| **Step 1:** Identify the leading terms in each expression.  | The leading terms are:$$3x^{4}$$ and $$2x^{4}$$ |
| **Step 2:** Combine the leading terms by the operation of addition. Recall that when adding or subtracting, exponent values remain the same. | $$3x^{4}+2x^{4}=5x^{4}$$ |
| **Step 3:** State the answer. | The leading term, or first term, of the solution will be $$5x^{4}$$. |

* Let’s examine how this process changes when doing multiplication or division.

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| **Example:** Determine the first term of the solution for the following operation. |
| **Step 1:** Identify the leading terms in each expression. | The leading terms are:$$x^{2}$$ and $$−9x^{2}$$ |
| **Step 2:** Combine the leading terms by the operation of multiplication. Recall the Product Rule of Exponents. |  |
| **Step 3:** State the answer. | The leading term, or first term, of the solution will be $$−9x^{4}$$. |

* To find the last term, examine the combinations of operations to determine what the lowest power of $$x $$

could be in the *polynomial* expression. Often, the lowest term is a constant that has no power.
* By determining the last term of the expression, you can determine the *polynomial* expression’s value when $$x=0 $$

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| **Example:** Determine the last term of the following expression.  |
| **Step 1:** Determine the lowest degree of each polynomial.  | Polynomial 1: $$7x\left(x^{2}+81\right)$$* Let’s simplify this expression so that we can understand more about the degrees of each term.
* $$7x\left(x^{2}+81\right)=7x^{3}+567x^{2}$$
* The lowest term is $$567x^{2}$$.

Polynomial 2: $$x^{2}$$* The lowest term is just $$x^{2}$$.
 |
| **Step 2:** Combine the lowest terms using addition as the operator, like the original expression.  | $$567x^{2}+x^{2}=568x^{2}$$ |
| **Step 3:** State the answer.  | The lowest term, or final term, of the solution will be $$568x^{2}$$. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Subtract the polynomial expressions and simplify completely.$$\left(3x^{5}+7x^{2}-4x+8\right)-\left(4x^{4}+8x^{2}-3x+7\right)$$ | $$3x^{5}-4x^{4}-x^{2}-x+1$$ |
| P 2 | Simplify the polynomial expression:$$\frac{7x^{2}+2x-9}{7x+9}+(x+2)(x-3)$$ | $$x^{2}-7$$ |
| P 3 | The length of a rectangle is $3x + 2$. Its area is $21x^{2}-x-10$. To find an expression that represents the width of the rectangle, perform the following operation(s) and simplify completely.$$\frac{21x^{2}-x-10}{3x+2}$$ | $$7x-5$$ |
| P 4 | Which of the following quotients can help identify the first term of the expression $\frac{22x^{3}+14x^{2}-3x}{2x+5}$?Option #1: $\frac{-3x}{2x}$Option #2: $\frac{22x^{3}}{2x}$Option #3: $\frac{14x^{2}}{5x}$ | 2 |
| P 5 | Identify the first and last terms of the simplified expression $4x^{2}\left(5x^{3}+4x+1\right)-9x^{5}+3x^{2}+4.$ | The first term is $11x^{5}$.The last term is 4. |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Perform the operation(s) and simplify completely.$$\left(7x^{2}-6x+2\right)-\left(4x-8\right)+(-6x^{2}+3x)$$ | $$x^{2}-7x+10$$ |
| Q 2 | Perform the operation(s) and simplify completely.$$(4-x)(\frac{-6x^{2}+10x+21}{4-2x}$$ | $$-6x^{2}+10x+21$$ |
| Q 3 | Perform the operation(s) and simplify completely.$$-\left(5x-2\right)\left(4x+8\right)+3x^{2}-9x$$ | $$-17x^{2}-41x+16$$ |
| Q 4 | Which of the following correctly determines the first and last terms of the expression $\left(12x^{5}+4x^{4}+9x^{3}-10x^{2}+15\right)-\left(24x^{5}+9x^{3}-7x^{2}+8\right)$? | The first term is $-12x^{5}$, and the last term is 7. |
| Q 5 | What is the highest degree for the expression $\left(3x^{2}+4\right)\left(x^{5}-3\right)-\frac{12x^{9}-24x}{3x+2}$ | 8 |

**Lesson 8 – Polynomial Identities**

**Key Words:**

* **polynomial identity –** a polynomial equation that is always true for any value of the variables

**Formulas:**

* Common Polynomial Identities:
	+ $$\left(x+a\right)^{2}=x^{2}+2ax+a^{2}$$
	+ $$\left(x+a\right)\left(x+b\right)=x^{2}+\left(a+b\right)x+ab$$
	+ $$x^{2}−a^{2}=\left(x+a\right)\left(x−a\right)$$
	+ $$x^{2}+a^{2}=\left(x+a\right)^{2}−2xa$$
	+ $$x^{3}−a^{3}=\left(x−a\right)\left(x^{2}+ax+a^{2}\right)$$
	+ $$x^{3}+a^{3}=\left(x+a\right)\left(x^{2}−ax+a^{2}\right)$$
	+ $$x^{4}−a^{4}=\left(x^{2}−a^{2}\right)\left(x^{2}+a^{2}\right)$$

**Objective 1:** In this section, you will divide polynomials by $$\left(x+a\right) or \left(x−a\right)$$

 to establish polynomial identities.

*Mathematical Practice Standard: Construct viable arguments and critique the reasoning of others.*

**Big Ideas**:

* Using *polynomial identities*, it is possible to factor expressions without performing multiplication or division.
* A *polynomial identity* is an equataion that is always true regardless of the values assigned to the variables. It is used to easily solve expressions involving larger exponents and numbers.
* *Polynomial identities* can be useful if it shows a pattern of calculation that is frequently called for. You can make such a calculation by substituting the same variables on both sides of the equation.
* *Polynomial identities* that fit a certain pattern can be used in two ways:
	+ 1. An equation that fits the pattern of a larger polynomial can be written as smaller factors without performing polynomial division.
	+ 2. A set of smaller factors that fits the pattern shown in the identity can be written as a larger polynomial without performing multiplication.

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| **Example #1:** Use the polynomial identity $$x^{2}+2ax+a^{2}=\left(x+a\right)^{2}$$ to rewrite $$x^{2}+16x+64$$. |
| **Step 1:** Match the terms up to determine what value *a* must be to fit the pattern. | What value of *a* will make both term 2 and 3 true?* Term 1: $$x^{2}=x^{2}$$
* Term 2: $$2ax=16x $$

$$2\left(8\right)x=16x$$* Term 3: $$a^{2}=64$$

$$\left(8\right)^{2}=64$$The value of *a* must be 8. |
| **Step 2:** Use the value of *a* to complete the identity. | $$x^{2}+2ax+a^{2}=\left(x+a\right)^{2}$$$$x^{2}+2\left(8\right)x+\left(8\right)^{2}=\left(x+8\right)^{2}$$ |
| **Step 3:** State the answer.  | The original polynomial $$x^{2}+16x+64$$ is equivalent to $$\left(x+8\right)^{2}$$. |

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| **Example #2:** Use the polynomial identity $$x^{2}+\left(a+b\right)x+ab=\left(x+a\right)\left(x+b\right)$$ to rewrite $$x^{2}+11x+24$$. |
| **Step 1:** Match the terms up to determine what value *a* and *b* must be to fit the pattern. | What values for *a* and *b* make term 2 and 3 true?* Term 1: $$x^{2}=x^{2}$$
* Term 2: $$\left(a+b\right)x=11x$$

$$\left(3+8\right)x=11x$$* Term 3: $$ab=24 $$

$$\left(3\right)\left(8\right)=24$$The values of *a* and *b* must be 3 and 8. |
| **Step 2:** Use the value of *a* and *b* to complete the identity. | $$x^{2}+\left(a+b\right)x+ab=\left(x+a\right)\left(x+b\right)$$$$x^{2}+\left(3+8\right)x+\left(3\right)\left(8\right)=\left(x+3\right)\left(x+8\right)$$ |
| **Step 3:** State the answer.  | The original polynomial $$x^{2}+11x+24$$ is equivalent to $$\left(x+3\right)\left(x+8\right)$$. |

**Objective 2:** In this section, you will use polynomial identities to find products by framing the multiplication as the difference of two squares.

*Mathematical Practice Standard: Look for and make use of structure.*

**Big Ideas**:

* Recall that p*olynomial identities* can be useful if it shows a pattern of calculation that is frequently called for. You can make such a calculation by substituting the same variables on both sides of the equation.
* The following *polynomial identities* can be used to convert differences of squares, trinomials, etc.
	+ 
* Using the *polynomial identity* of the difference of two squares is convenient because it reduces multiplication to simple subtraction.
	+ Examples using the identity $$x^{2}−a^{2}=\left(x+a\right)\left(x−a\right)$$

	 when terms are perfect squares.
		- $$x^{2}−8^{2}=\left(x+8\right)\left(x−8\right)$$
		- $$x^{2}−3^{2}=\left(x+3\right)\left(x−3\right)$$
	+ Examples using the identity $$x^{3}−a^{3}=\left(x−a\right)\left(x^{2}+ax+a^{2}\right)$$

	 when terms are perfect cubes.
		- $$x^{3}−5^{3}=\left(x−5\right)\left(x^{2}+5x+5^{2}\right)$$
		- $$x^{3}−4^{3}=\left(x−4\right)\left(x^{2}+4x+4^{2}\right)$$
* Let’s work through a more complicated example:

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| **Example:** Convert the difference of cubes into a product. $$x^{6}−125$$ |
| **Step 1:** Identify the two perfect cubes.  | * Term 1: $$x^{6}=\left(x^{2}\right)^{3}$$
* Term 2: $$125=\left(5\right)^{3}$$

$$\left(x^{2}\right)^{3}−5^{3}$$ |
| **Step 2:** Use the identity to rewrite the expression as a product. | The identity for the difference of cubes is:$$x^{3}−a^{3}=\left(x−a\right)\left(x^{2}+ax+a^{2}\right)$$For every *x* plug in $$x^{2}$$. For every *a* plug in 5.$$\left(x^{2}\right)^{3}−5^{3}=\left(x^{2}−5\right)\left(\left(x^{2}\right)^{2}+5\left(x^{2}\right)+5^{2}\right)$$ |
| **Step 4:** Simplify the expression on the right side.  | $$\left(x^{2}−5\right)\left(\left(x^{2}\right)^{2}+5\left(x^{2}\right)+5^{2}\right)$$$$\left(x^{2}−5\right)\left(x^{4}+5x^{2}+25\right)$$ |
| **Step 5:** State the answer.  | $$x^{6}−125$$ converted into a product is $$\left(x^{2}−5\right)\left(x^{4}+5x^{2}+25\right)$$.$$x^{6}−125=\left(x^{2}−5\right)\left(x^{4}+5x^{2}+25\right)$$ |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Which of the following polynomial identities would result from dividing $x^{3}+a^{3}$ by $x+a$? Enter the option number of the correct answer.Option #1: $x^{2}+a^{2}$Option #2: $x^{2}-ax+a^{2}$Option #3: $x^{2}+ax+a^{2}$ | 2 |
| P 2 | Which of the following polynomial identities would result from dividing $x^{2}-a^{2}$ by $x-a$? Enter the option number of the correct answer.Option #1: $x^{2}-a^{2}=(x-a)(x+a)$Option #2: $x^{2}-a^{2}=(x-a)(x-a)$Option #3: $x^{2}-a^{2}=(x+a)(x+a)$ | 1 |
| P 3 | Which of the following polynomial identities would result from dividing $x^{2}+\left(a+b\right)x+ab$ by $x+a$? Enter the option number of the correct answer.Option #1: $x^{2}+\left(a+b\right)x+ab=(x+a)(x-b)$Option #2: $x^{2}+\left(a+b\right)x+ab=(x-a)(x+b)$Option #3: $x^{2}+\left(a+b\right)x+ab=(x+a)(x+b)$ | 3 |
| P 4 | How would you use the difference of two squares identity to multiply two numbers that are equidistant from another number? Enter the number of the correct option. Option #1: Use the equation $x^{2}-a^{2}$, where *a* is the middle number (midpoint) and *x* is the distance from the midpoint to either endpoint. Option #2: Use the equation $x^{2}-a^{2}$ , where *x* is the middle number (midpoint) and *a* is the distance from the midpoint to either endpoint. Option #3: This operation is not possible without more information. | 2 |
| P 5 | Use the polynomial identity of the difference of two squares to write a product equal to $81-16$. | $$(9-4)(9+4)$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Divide the polynomial $x^{3}-a^{3}$ by $x-a$. Which polynomial identity does this establish? | $$x^{3}-a^{3}=(x-a)(x^{2}+ax+a^{2}$$ |
| Q 2 | Use the polynomial identity $x^{2}+2ax+a^{2}=(x+a)(x+a)$ to rewrite $x^{2}+4x+4$. Which polynomial does this establish? | $$x^{2}+4x+4=(x+2)(x+2)$$ |
| Q 3 | Use the difference of two squares to find the product 4,250 and 5,750. | open parentheses 5 comma 000 plus 750 close parentheses open parentheses 5 comma 000 minus 750 close parentheses equals 24 comma 437 comma 500 {"mathml":"<math style=\"font-family:stix;font-size:12px;\" xmlns=\"http://www.w3.org/1998/Math/MathML\"><mstyle mathsize=\"12px\"><mfenced><mrow><mn>5</mn><mo>,</mo><mn>000</mn><mo>+</mo><mn>750</mn></mrow></mfenced><mfenced><mrow><mn>5</mn><mo>,</mo><mn>000</mn><mo>-</mo><mn>750</mn></mrow></mfenced><mo>=</mo><mn>24</mn><mo>,</mo><mn>437</mn><mo>,</mo><mn>500</mn></mstyle></math>","origin":"MathType for Microsoft Add-in"}, Picture |
| Q 4 | How can 13 × 19 be rewritten using the difference of two squares identity? | $$(16 - 3) (16 + 3)$$ |
| Q 5 | Hiram has a blanket measuring 55 inches by 43 inches. Which of the following expressions can he use to find the area of the blanket? | $$49^{2}-6^{2}$$ |

**Lesson 9 – Polynomial Power**

**Key Words:**

* **composite number** – a number with factors besides itself and 1
* **Mersenne prime** – a prime number that is one less than some power of 2
* **polynomial identity** – a polynomial equation that is always true for any value of the variables
* **prime number** – a number whose only factors are itself and 1
* **Pythagorean triple** – a set of three positive integers such that the sum of the squares of the first two numbers equals the square of the third number

**Formulas:**

* Pythagorean Theorem:$$a^{2}+b^{2}=c^{2}$$
* Mersenne prime:$$2^{n}−1$$
* Pythagorean triple polynomial identity: $$\left(x^{2}+y^{2}\right)^{2}=\left(x^{2}−y^{2}\right)^{2}+\left(2xy\right)^{2}$$

**Objective 1:** In this section, you will use polynomial identities to find prime numbers.

*Mathematical Practice Standard: Reason abstractly and quantitatively.*

**Big Ideas**:

* Recall that *prime numbers* have no factors besides themselves and 1.
	+ For example: 3, 5, 7, etc.
* A special category of primes is called *Mersenne primes*.
	+ These primes consist of primes that are 1 less than some power of 2, or, prime values of $$p $$

	where $$p=2^{n}−1$$

	 .
	+ For example, 7 is a Mersenne prime since it equals $$2^{3}−1$$

	.
* *Polynomial identities* can help find *Mersenne primes*. The trick to finding Mersenne primes is to determine what values of $$n $$

could generate a prime in the form of $$2^{n}−1$$

.
	+ If $$2^{n}−1$$

	 is prime, $$n $$

	must be prime.
	+ If $$n $$

	is prime, $$2^{n}−1$$

	 could be prime or composite.
	+ Because of this relationship, each must be tested separately to see if $$2^{n}−1$$

	 is also prime.

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| **Determine if the following values are prime.** |
| Value | Reason | Prime? |
| $$2^{11}−1$$ | $$n =11 $$, $$n $$is a prime number but it must be tested to see if $$2^{11}−1 $$is prime. $$2^{11}−1 =2047$$2047 is not a prime number. | No |
| $$2^{12}−1$$ | $$n=12 $$, $$n $$is not a prime number, therefore $$2^{12}−1$$ is not a prime number. | No |
| $$2^{13}−1$$ | $$n=13 $$, $$n $$ is a prime number but it must be tested to see if $$2^{13}−1$$ is prime.$$2^{13}−1=8191$$8191 is a prime number.  | Yes |
| $$2^{14}−1$$ | $$n=14 $$, $$n $$ is not a prime number, therefore $$2^{14}−1$$ is not a prime number. | No |
| $$2^{19}−1$$ | $$n=19 $$, $$n $$ is a prime number but it must be tested to see if $$2^{19}−1$$ is prime.$$2^{19}−1=524,287$$$$524,287 $$is a prime number. | Yes |

**Objective 2:** In this section, you will learn how to generate Pythagorean triples using a polynomial identity.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas:**

* A *Pythagorean triple* is a set of three positive integers such that the squares of the first two integers add up to the square of the third.
	+ If a, b, and c are a *Pythagorean triple*, then $$a^{2}+b^{2}=c^{2}$$

	, also written as $$c^{2}=a^{2}+b^{2}$$

	.
	+ For example, a common *Pythagorean triple* is {3,4,5}.
		- Let *a=3*, *b=4*, and *c=5*.
		- $$3^{2}+4^{2}=5^{2}$$
		- $$9+16=25 $$
		- $$25=25 $$
* A *polynomial identity* makes it possible to find *Pythagorean triples* by providing a pattern that you can use and reuse with different values.

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| The *polynomial identity* $$\left(x^{2}+y^{2}\right)^{2}=\left(x^{2}−y^{2}\right)^{2}+\left(2xy\right)^{2}$$ fits the pattern of the Pythagorean theorem, $$c^{2}=a^{2}+b^{2}$$.* all three terms are squared
* two of the squared terms add to the third squared term

The three expressions inside the parentheses of the identity $$\left(x^{2}+y^{2}\right)^{2}=\left(x^{2}−y^{2}\right)^{2}+\left(2xy\right)^{2}$$ can be taken as the *c*, *a*, and *b* of a *Pythagorean triple*. * $$c=x^{2}+y^{2}$$
* $$a=x^{2}−y^{2}$$
* $$b=2xy^{}$$
 |

* This identity can be used to generate *Pythagorean triples* by substituting various integer values for *x* and *y*. There are two conditions to consider:
	+ *x* and *y* must be positive, since the length is never negative.
	+ *x* must be greater than *y*, or else $$x^{2}−y^{2}$$

	 would not be positive.

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| **Example:** Use the polynomial identity $$\left(x^{2}+y^{2}\right)^{2}=\left(x^{2}−y^{2}\right)^{2}+\left(2xy\right)^{2}$$ to generate a Pythagorean triple when x equals 5 and *y* equals 3. |
| **Step 1:** Substitute the *x* and *y* values into the identity. | $$\left(x^{2}+y^{2}\right)^{2}=\left(x^{2}−y^{2}\right)+\left(2xy\right)^{2}$$Recall that each term in the parentheses represents *c*, *b* and *a*.* $$c=x^{2}+y^{2}$$

$$c=5^{2}+3^{2}=25+9=34$$* $$a=x^{2}−y^{2}$$

$$a=5^{2}−3^{2}=25−9=16$$* $$b=2xy $$

$$b=2\left(5\right)\left(3\right)=30$$ |
| **Step 3:** Test the Pythagorean triple by substituting the values for *c*, *b*, and *a* into the Pythagorean theorem. | $$c^{2}=a^{2}+b^{2}$$$$c=34, a=16, b=30 $$$$34^{2}=16^{2}+30^{2}$$$$1156=256+900 $$$$1156=1156 $$ |
| **Step 4:** State the answer. | $$34^{2}$$ is equal to the sum of $$16^{2}$$and $$30^{2}$$, thus, {16,30,34} is a Pythagorean triple. |

**Practice Questions and Answers**

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| --- | --- | --- |
|  | Question | Answer |
| P 1 | Which of the following expressions could, by definition, yield a Mersenne prime number? Enter the number of the correct option.Option #1: $2^{9}-1$Option #2: $2^{11}-1$Option #3: $2^{15}-1$ | 2 |
| P 2 | Which of the following expressions can be used to determine a Mersenne prime? Enter the number of the correct option.Option #1: $2^{n}+1$Option #2: $2^{n}-1$Option #3: $2n-1$ | 2 |
| P 3 | Which of the following statements must be true about n in the expression for a Mersenne prime,$2^{n}-1$? Enter the number of the correct statement. Statement #1: It must be an even number. Statement #2: It must be an odd composite number. Statement #3: It must be a prime number | 3 |
| P 4 | Mei wants to generate a Pythagorean triple when *x = 8* and *y* = 6 using the polynomial identity $\left(x^{2}+y^{2}\right)^{2}= \left(x^{2}-y^{2}\right)^{2}+(2xy)^{2}$ . What are the three values of the Pythagorean triple from smallest to largest? | {28, 96, 100} is a Pythagorean triple. |
| P 5 | Use the polynomial identity $\left(x^{2}+y^{2}\right)^{2}= \left(x^{2}-y^{2}\right)^{2}+(2xy)^{2}$ to generate a Pythagorean triple when *x* = 7 and *y* = 4. Write the Pythagorean triple from smallest value to largest value. | {33, 56, 65} is a Pythagorean triple. |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which of the following properly uses a polynomial to identity to detect if $2^{2k}-1$ is prime? | $2^{2k}-1=(2^{k}+1)(2^{k}-1)$, so $2^{2k}-1$ is not prime. |
| Q 2 | Which of the following numbers is a prime in the form $2^{n}-1$? | 127 |
| Q 3 | Ogechi is trying to demonstrate that 7 is a Mersenne prime. Which of the following expressions should she use? | $$2^{3}-1$$ |
| Q 4 | Use the polynomial identity $\left(x^{2}+y^{2}\right)^{2}= \left(x^{2}-y^{2}\right)^{2}+(2xy)^{2}$ to generate a Pythagorean triple when *x* equals 7 and *y* equals 3. Which of the following is one of the Pythagorean triple? | 40 |
| Q 5 | Matilde wants to cut a right triangle for a quilt from a piece of fabric. One side of the piece of fabric is 16 centimeters long. To make measuring easier, Matilde wants the other two sides of the triangle to be whole numbers of centimeters. Use the polynomial identity $$\left(x^{2}+y^{2}\right)^{2}=\left(x^{2}−y^{2}\right)^{2}+2xy$$ to find the other possible side lengths, assuming $$2xy=16 $$and that the two undetermined sides are of different lengths. Which of the following values could be the other two side lengths for the triangle? | 12 cm and 20 cm |

**Lesson 10 – The Binomial Theorem**

**Key Words:**

* **binomial expansion** – the complete set of terms of a binomial raised to a higher power
* **Binomial Theorem** – a theorem that describes the full expansion of a binomial raised to a higher power
* **Pascal’s Triangle** – an arrangement of numbers in increasing rows such that the first and last numbers in each row are 1, and the numbers in between equal the sum of the two numbers above them

**Formulas:**

* Binomial Theorem:
	+ Series: 
	+ Combination: 
* Combination: $$\_{n}C\_{r}$$

 is the number of combinations of *n* objects taken *r* at a time.
* Factorial: $$n! $$

**Objective 1:** In this section, you will learn how Pascal’s Triangle relates to the expansion of binomials in the form of $(a+b)^{n}$.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas**:

* *Binomial expansion* is the complete set of a binomial raised to a higher power.
	+ For example, the binomial expansion of $$\left(a+b\right)^{2}$$

	 is $$a^{2}+2ab+b^{2}$$

	. This is achieved by multiplying $$\left(a+b\right)\left(a+b\right)$$

	.
* What if you wanted to find the *binomial expansion* of $$\left(a+b\right)^{8}$$

 or $$\left(a+b\right)^{20}$$

? This would be difficult to find by doing traditional multiplication.
* You can use *Pascal’s Triangle* to make *binomial expansion* more manageable.

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| **Pascal’s Triangle** |
| * *Pascal’s Triangle* is an arrangement of numbers in a triangle with two rules forming it.
	+ 1. The first and last number in each row is 1.
	+ 2. Each intervening number is the sum of the two numbers above it.
	+ Note that you can add as many rows to the triangle as needed, following the above rules.

 |

* Let’s examine the pattern that *Pascal’s triangle* provides. Notice that the numbers in each row correspond to the coefficients of the *binomial expansion*.

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| **Binomial** | **Row** | **Binomial Expansion** |
| $$\left(a+b\right)^{1}$$ |  | $$a+b $$ |
| $$\left(a+b\right)^{2}$$ |  | $$1a^{2}+2ab+1b^{2}$$ |
| $$\left(a+b\right)^{3}$$ |  | $$1a^{3}+3a^{2}b+3ab^{2}+1b^{3}$$ |

* Now that you know how to find the coefficients of the expansion, you must consider the exponents.
* When expanding $$\left(a+b\right)^{n}$$

:
	+ the exponents for *a* start at *n* and decrease to zero
	+ the exponents for *b* start at zero and increase up to *n*
* Let’s apply this rule to$$\left(a+b\right)^{2}$$

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| $$\left(a+b\right)^{2}$$* Use *Pascal’s Triangle* to identify the coefficients.
* The exponents for *a* start at 2 and decrease to 0.
* The exponents for *b* start at 0 and increase up to 2.

* From here, we can simplify the expression.
* Recall that any number raised to the power of 0 is 1.
* Recall that a number or variable raised to the power of 1 is itself.

$$1a^{2}b^{0}+2a^{1}b^{1}+1a^{0}b^{2}=1a^{2}+2ab+1b^{2}$$* Notice that each term's exponents add to *n*.
 |

**Objective 2:** In this section, you will use the Binomial Theorem to raise binomials like $(a+b)$ to whatever power you choose.

*Mathematical Practice Standard: Construct viable arguments and critique the reasoning of others.*

**Big Ideas:**

* So far, we have learned that we can carry out a *binomial expansion* by multiplying the binomial by itself or using *Pascal’s Triangle*. These tools can become difficult to use as binomial exponents become larger.
* There is a direct connection between the *Binomial Theorem* and the mathematics of permutations and combinations.
	+ Combinations, written as $$\_{n}C\_{r}$$

	, describe the number of unordered sets of *r* items you could make out of a total store of *n* items.
		- You can use a scientific calculator; you can find the coefficients for a *binomial expansion* by using the calculator’s $$\_{n}C\_{r}$$

		 feature.
		- For example, $$\_{7}C\_{5}$$

		 is the number of different five-person basketball teams you could make from a roster of seven available players. [Calculate in GeoGebra](https://www.geogebra.org/calculator/nfm49jwc).
* The *Binomial Theorem* is the most comprehensive method of *binomial expansion*.
	+ It allows you to calculate the coefficient of each term and the exponents of all the variables.

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| **Binomial Theorem** |
| * The fractional part of the formula computes the coefficient.
* The notation $$n! $$means $$n− $$factorial, the product of every integer from $$n $$down to 1.
* The exponents of *a* will decrease and the exponents of *b* will increase as *k* progresses from 0 to $$n $$.

The Binomial Theorem can also be written as a combination: |

* Recall that $$n! $$

represents a factorial.
	+ $$4!=4⋅3⋅2⋅1=24 $$
	+ $$7!=7⋅6⋅5⋅4⋅3⋅2⋅1=5040 $$
	+ Note that $$0!=1 $$

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| **Example:** Use the Binomial Theorem to expand $$\left(a+b\right)^{4}$$. |
| **Step 1:** Start with the general statement of the Binomial Theorem.  | $$\left(a+b\right)^{n}=\sum\_{k=0}^{n}\frac{n!}{\left(n−k\right)!k!}a^{n−k}b^{k}$$ |
| **Step 2:** Set *n* to the power of the binomial and substitute into the formula.  | $$\left(a+b\right)^{4}=\sum\_{k=0}^{4}\frac{4!}{\left(4−k\right)!k!}a^{4−k}b^{k}$$ |
| **Step 3:** There will be a term for every value of *k* from 0 to *n*.  | There will be five terms in all to represent$$k=0,1,2,3,4 $$**.** Write out the five terms, replacing *k* in each term with the appropriate value: |
| **Step 4:** Remove all the parentheses, and simplify the exponents of *a.* |  |
| **Step 5:** Simplify. A variable with an exponent of 0 is equal to 1 and can be omitted entirely. A variable with an exponent of 1 is simply the variable.  |  |
| **Step 6:** Compute the coefficients (the fractional part of each term).  | $$4!=4⋅3⋅2⋅1=24 $$$$3!=3⋅2⋅1=6 $$$$2!=2⋅1=2 $$$$1!=1 $$$$0!=1 $$ |
| **Step 7:** Simplify. |  |
| **Step 8:** Omit the coefficients 1 in the first and last terms. |  |

* Let’s try an example using the combinations form of the formula.

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| **Example:** Use the combinations form of the Binomial Theorem to expand $$\left(a+b\right)^{5}$$. |
| **Step 1:** Start with the combinations form of the Binomial Theorem. | $$\left(a+b\right)^{n}=a^{n}+\_{n}C\_{1}a^{n−1}b+\_{n}C\_{2}a^{n−2}b^{2}+…+\_{n}C\_{n−1}ab^{n−1}+b^{n}$$ |
| **Step 2:** Identify the value of *n* and plug it into the formula and simplify each term.  |  |
| **Step 3**: Use a scientific calculator to calculate each coefficient.  | [GeoGebra calculations linked here.](https://www.geogebra.org/calculator/fxdasu3d) $$\_{5}C\_{0}=1$$$$\_{5}C\_{1}=5$$$$\_{5}C\_{2}=10$$$$\_{5}C\_{3}=10$$$$\_{5}C\_{4}=5$$$$\_{5}C\_{5}=1$$ |
| **Step 4:** Plug the coefficients into the formula.  | $$a^{5}+5a^{4}b+10a^{3}b^{2}+10a^{2}b^{3}+5ab^{4}+b^{5}$$ |

**Objective 3:** In this section, you will apply the Binomial Theorem to write out the full expansion of binomials raised to higher powers.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

 **Big Ideas:**

* The *Binomial Theorem* works the same with more complex binomials as it does for $$a+b $$

.
	+ For example: $$8x+9y $$
	+ $$8x $$

	represents the entire first term and $$9y $$

	represents the second term.
* Using the various methods of binomial expansion – Pascal's Triangle, Binomial Theorem, and combinations – let's work through more complex examples.

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| **Example:** Find the binomial expansion of $$\left(7x+6y\right)^{3}$$. |
| **Step 1:** Find the base binomial that the expression represents.  | $$\left(7x+6y\right)^{3}$$ follows the pattern of $$\left(a+b\right)^{3}$$.$$a=7x $$$$b=6y $$ |
| **Step 2:** Use a method of binomial expansion to show $$\left(a+b\right)^{3}$$. | $$\left(a+b\right)^{3}=1a^{3}+3a^{2}b+3ab^{2}+1b^{3}$$ |
| **Step 3:** Replace *a* and *b* with the terms of the original binomial. | $$\left(7x+6y\right)^{3}=1\left(7x\right)^{3}+3\left(7x\right)^{2}\left(6y\right)+3\left(7x\right)\left(6y\right)^{2}+1\left(6y\right)^{3}$$ |
| **Step 4:** Use the Commutative Property of Multiplication, collect all the coefficients together and all the variables together for each term. |  |
| **Step 5:** Simplify. | $$\left(7x+6y\right)^{3}=343x^{3}+882x^{2}y+756xy^{2}+216y^{3}$$ |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Using Pascal’s Triangle, write out the expansion of $(a+1)^{3}$. | $$a^{3}+3a^{2}+3a+1$$ |
| P 2 | Using Pascal’s Triangle, what is the coefficient of the third term in the expansion of $(a+b)^{6} $when the expanded polynomial is written in standard form? | 15 |
| P 3 | Write out the binomial expansion of $(a+b)^{4}$ using the Binomial Theorem. | $$a^{4}+4a^{3}b+6a^{2}b^{2}+4ab^{3}+b^{4}$$ |
| P 4 | Using the Binomial Theorem, what would be the coefficient of the sixth term in the binomial expansion of $(a+b)^{9}$? | 126 |
| P 5 | If $5x-7$ is raised to the fifth power, which terms will be negative? Enter the option number of the correct answer.Option 1: the first, third, and fifth terms Option 2: the second, fourth, and sixth terms Option 3: It is impossible to tell without performing the binomial expansion. | 2 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Use Pascal’s Triangle to write out the expansion of $(a+b)^{3}$. | $$a^{3}+3a^{2}b+3ab^{2}+b^{3}$$ |
| Q 2 | Using Pascal’s Triangle, what is the coefficient of the second term in the expanded form of $(a+3)^{7}$ when the expanded polynomial is written in standard form? | 21 |
| Q 3 | Which of the following establishes the Binomial Theorem for the expansion of polynomials in the form $(a+b)^{n}$. |  |
| Q 4 | Geoffrey writes out the binomial expansion of $(a+b)^{6}$ using the Binomial Theorem. Which of the following can he use to find the coefficient for the $a^{4}b^{2}$ term? | 6C2 |
| Q 5 | Jaime applies the Binomial Theorem for the expansion of $(x-5)^{4}$. Which of the following is the $x^{2}$ term? | $$150x^{2}$$ |