Algebra 1

**Linear & Exponential Sequences**

**Unit Summary:** In this unit, you will explore how arithmetic sequences related to linear functions and how geometric sequences relate to exponential functions. You will graph sequences and learn how to write functions to represent sequences using recursive and explicit formulas. Finally, you will use sequences and graphs to compare the growth rates of linear functions and exponential functions.

**Lesson 2 – Introduction to Sequences**

**Key Words:**

* **arithmetic sequence** – an ordered list of terms in which the difference between any two consecutive terms remains constant
* **common difference** – in an arithmetic sequence, the value of the difference between any two consecutive terms
* **common ratio** – in a geometric sequence, the value of the ratio between any two consecutive terms
* **domain** – the set of all possible inputs (x-values) of a function
* **function** – an expression, rule, or law that defines a relationship between the independent variable and the dependent variable
* **geometric sequence** – an ordered list of terms in which the ratio between any two consecutive terms remains constant
* **index** – a natural (counting) number that indicates the position of a term in a sequence, starting with 1 or 0
* **notation** – a system of characters, symbols, or abbreviated expressions used in art, science, mathematics, or logic to express technical facts or quantities
* **quadratic sequence** – a sequence in which the terms are the output of a quadratic expression, that is, in which the index is raised to the second power and no higher
* **range** – the set of all outputs (y-values) of a function
* **recursion** – the determination of a succession of elements (such as numbers or functions) by operation on one or more preceding elements according to a rule or formula involving a finite number of steps
* **recursive formula** – a formula expressing any term of a sequence as a function of one or more preceding terms
* **sequence** – a set of numbers that follow a specific pattern or formula
* **subscript** – a distinguishing symbol (such as a letter or numeral) written immediately below or below and to the right or left of another character
* **term** – is either a single number or variable, or numbers and variables multiplied together; can be used to refer to the individual items in a sequence of values

**Objective 1:** In this section, you willdetermine the values of terms in a given sequence.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas**:

* A *sequence* is any list of numbers in some given order.
  + The numbers in the sequence are called *terms*.
  + The position of a term in the sequence is called its *index*. For example, the fifth number in a sequence has an *index* of 5.
* If you are given a *sequence* of numbers, you can determine the difference between pairs of adjacent terms by subtracting each *term* from the *term* after it.
  + If the **difference** is constant, it is an *arithmetic sequence*.
  + If the **ratio** is constant, it is a *geometric sequence*.

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| **Arithmetic Sequence** | **Geometric Sequence** |
| * An *arithmetic sequence* is a list of *terms* (numbers) that progress by **adding or subtracting** each term by the same amount each time, called the *common difference*. * For example, the following arithmetic sequence has a *common difference* of +3.      * The rule to determine the next term is to add the *common difference*. | * A *geometric sequence* is a list of *terms* (numbers) that progress by **multiplying or dividing** each term by the same amount each time, called the *common ratio*. * For example, the following geometric sequence has a *common ratio* of x2.      * The rule to determine the next term is to multiply by the *common ratio*. |

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| **Example: Arithmetic Sequence**  For the following arithmetic sequence, determine:   1. the rule for generating the next term 2. the next three terms   24, 19, 14, 9, ... | |
| **Step 1:** Determine the common difference between any term and its preceding term. | Any of these will work:  1st and 2nd term:  3rd and 4th term:  The common difference of the arithmetic sequence is –5. |
| **Step 2:** Determine the next three terms by adding the common difference to the preceding term. Making a table can be helpful. | |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | Index (*n*) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | Rule | given | given | given | given |  |  |  | | *n*th term | 24 | 19 | 14 | 9 | 4 | -1 | -6 | |

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| **Example: Geometric Sequence**  For the following geometric sequence, determine:   1. the rule for generating the next term 2. the next three terms   81, 54, 36, 24, ... | |
| **Step 1:** Determine the common ratio by dividing any term by its preceding term. | Any of these will work:  1st and 2nd term:  3rd and 4th term:  The common ratio of the geometric sequence is. |
| **Step 2:** Determine the next three terms by multiplying the common ratio by the preceding term. Making a table can be helpful. | |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | Index (*n*) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | Rule | given | given | given | given |  |  |  | | *n*th term | 81 | 54 | 36 | 24 | 16 |  |  | |

**Objective 2:** In this section, you will classify a list of numbers by the type of rule that generates it, either as an arithmetic sequence, a geometric sequence, another type of sequence, or not a known sequence.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas:**

* Recall that a *common difference* is found by subtracting a term from the previous term.
* Recall that a *common ratio* is found by dividing a term by the previous term.
* Recall that a table is a good way to organize and represent calculations of a sequence. The table should include the index, term, and difference/ratio.

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| **Sequence** | **How to Identify** | **Example** |
| **Arithmetic**   * have a common difference between each term in the sequence and the previous one | * Subtract each term from the next term. * If the difference between each term and the term before it is the same for all pairs of consecutive terms, then the list represents an arithmetic sequence. |  |
| **Geometric**   * have a common ratio between each term in the sequence and the previous one | * Divide each term by the term before it. * If the ratio between each term and the term before it is the same for all pairs of consecutive terms, then the list represents a geometric sequence. |  |
| **Quadratic**   * provide a list of differences between consecutive terms that is itself an arithmetic sequence | * Subtract each term from the next term to find the first difference, as you would for an arithmetic sequence. * Find the common differences of the first differences found, called the second difference. |  |

* To classify a list of numbers as a certain class of sequence, you can apply the previous tests on every pair of consecutive terms. If all the pairs fit the pattern, then the sequence is in that class.

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| **Example:** Consider the following list of numbers and classify them as arithmetic, geometric, or a quadratic sequence. | |
| **Step 1:** Calculate the first difference. | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | **Index (*n*)** | 1 | 2 | 3 | 4 | 5 | | ***n*th term** | 1.5 | -3 | 6 | -12 | 24 | | **difference** | - | -4.5 | 9 | -18 | 36 |   The differences are all different, this is not an arithmetic sequence. |
| **Step 2:** Calculate the second difference. | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | **Index (*n*)** | 1 | 2 | 3 | 4 | 5 | | ***n*th term** | 1.5 | -3 | 6 | -12 | 24 | | **difference** | - | -4.5 | 9 | -18 | 36 | | **2nd difference** | - | - | 13.5 | -27 | 54 |   The 2nd differences are all different, this is not a quadratic sequence. |
| **Step 3:** Calculate the ratios. | |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | | **Index (*n*)** | 1 | 2 | 3 | 4 | 5 | | ***n*th term** | 1.5 | -3 | 6 | -12 | 24 | | **Ratio** | - | -2 | -2 | -2 | -2 |   The ratios are all the same, -2, therefore this is a geometric sequence. |

**Objective 3:** In this section, you will use subscript notation to describe sequences and terms, and you will relate this notation to function notation.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas:**

* Recall that each term in a *sequence* is assigned an *index*, or position, which is given the variable letter *n*.
* Two concise ways to refer to any specific term in a *sequence* are *function notation* and *subscript notation*.

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| **Function Notation**, *f(n)*: where *f* is the identifier for the sequence and the expression in the parentheses is the index, or position of the number in the sequence. |  |
| **Subscript Notation**, : where *a* is the identifier for the sequence and the subscript is the index, or position of the number in the sequence. |
| **Example:** Given the following sequence, write an expression in function notation and subscript form for the fourth term in the sequence.  The fourth term in the sequence is 18.  Function Notation:  Subscript Notation: | |

* Some sequences can be defined by a rule or formula that tells you how to determine the *n*th term.
* If the only variable in the formula is the index *n*, then the formula is called an *explicit formula*.

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| **Example:** Write a formula for a sequence in which the nth term equals the **index multiplied by 2.5** **then decreased by 3**, in both function notation and subscript notation. Use one of the forms to find the 7th term. | |
| **Step 1:** Write the explicit formulas. |  |
| **Step 2:** Evaluate one of the formulas for the 7th term. | The 7th term in the sequence is 14.5. |

* Some terms in a sequence are defined using a previous term. When other terms are defined in terms of a previous term or terms, you will use a *recursive formula*.

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| **Example:** If the *n*th term of a sequence can be written as or *f(n)*, what is the term just before the *n*th term? |
| Since the natural number before *n* is *(n-1)*, the term before the *n*th term is . |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Which of the following statements correctly describes the sequence 3, 8, 13, 18, . . .?  Statement #1: It is an arithmetic sequence.  Statement #2: It is a geometric sequence.  Statement #\_\_\_\_ correctly describes the sequence. | 1 |
| P 2 | Classify the following set of numbers by determining which of the options is true.  3, 1, , , , . . .  Option #1: The set is an arithmetic sequence.  Option #2: The set is a geometric sequence.  Option #3: The set is some other type of sequence.  Option #4: The set is not a sequence.  The true statement is Option #\_\_\_\_\_. | 2 |
| P 3 | What is the common difference in the following arithmetic sequence?  , …..  The common difference in the arithmetic sequence is \_\_\_\_\_. |  |
| P 4 | Given the first five terms of the sequence 3, 7, 11, 15, 19 . . . determine which option is the correct subscript notation for the explicit formula.  Option #1:  Option #2:  Option #3:  Option #\_\_\_\_\_ | 3 |
| P 5 | Given the first five terms of the sequence  determine which option is the correct subscript notation for the recursive formula.  Option #1: ;  Option #2: ;  Option #3: ;  Option #\_\_\_\_ | 1 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which of the following statements correctly determines the rule for generating the next term in the sequence 23, 19.5, 16, 12.5, . . .? | Add the common difference of −3.5. |
| Q 2 | Classify the following list of numbers as an arithmetic sequence, a geometric sequence, some other sequence, or not a sequence.  9, 4, −1, −6, . . . | arithmetic sequence |
| Q 3 | What is the common ratio in the following geometric sequence? |  |
| Q 4 | Given the first five terms of the sequence 19, 15, 11, 7, 3 . . . , use subscript notation to describe . |  |
| Q 5 | Given the recursively defined formula , determine which option is the correct function notation. | ; |

**Lesson 3 – Formulas for Sequences**

**Key Words:**

* **arithmetic sequence** – an ordered list of terms in which the difference between any two consecutive terms remains constant
* **explicit formula** – a formula to determine the nth term of different types of sequences (e.g., )?
* **geometric sequence** – an ordered list of terms in which the ratio between any two consecutive terms remains constant
* **recursive formula** – a formula expressing any term of a sequence as a function of one or more preceding terms

**Formulas:**

* Arithmetic
  + Explicit:
  + Recursive: ,
* Geometric
  + Explicit:
  + Recursive: ,

**Objective 1:** In this section, you will use recursive and explicit formulas to write sequences.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* Recall that *explicit formulas* allow you to find any term in a sequence.
* Recall that *recursive formulas* help you find a term by using the previous term.
* Use the following table to recall the formulas for each type of sequence, *arithmetic* and *geometric*.

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|  | **Explicit** | **Recursive** |
| **Arithmetic**  *d = common difference* |  |  |
| **Geometric**  *r = common ratio* |  |  |

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| **Example 1 (explicit):** Given the **explicit** formula, find the first five terms. | |
| **Step 1:** Plug in 1,2,3,4, and 5 for *n.* It’s best to use a table to organize your work. |  |
| **Step 2:** Use order of operations to solve the equations with the substituted value. The results will be the terms of the sequence. |
| **Step 3:** Write the sequence. | The first five terms of the sequence are: |

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| **Example 2 (recursive):** Given the **recursive** formula, find the first five terms using a table. | |
| **Step 1:** Plug in 1,2,3,4, and 5 for *n.* It’s best to use a table to organize your work. |  |
| **Step 2:** Use order of operations to solve the equations with the substituted value. The results will be the terms of the sequence. |
| **Step 3:** Write the sequence. | The first five terms of the sequence are: |

**Objective 2:** In this section, you will use recursive formulas for arithmetic and geometric sequences.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* [Recall](#Bookmark1) the *recursive formulas* for arithmetic and geometric sequences.
* *Arithmetic sequences* represent linear growth. The *recursive formula* for *arithmetic sequences* relies on the repeated addition (*common difference*) to the previous term to arrive at the current term.

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| **Example:** Given the following arithmetic sequence, find the recursive formula. | | |
| **Step 1:** Find the first term in the sequence. | The first term is given and is 19.3. | |
| **Step 2:** Find the common difference. | Recall that the common difference is found by subtracting one term from a previous term. You can use any two consecutive terms.  1st and 2nd term: | |
| **Step 3:** Write the recursive formula. | Subscript Notation: | Function Notation: |

* *Geometric sequences* represent exponential growth or decay. The *recursive formula* for *geometric sequences* uses the *common ratio* and the previous term of the sequence.

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| **Example:** Given the following geometric sequence, find the recursive formula. | | |
| **Step 1:** Find the first term in the sequence. | The first term is given and is -100. | |
| **Step 2:** Find the common ratio. | Recall that the common ratio is found by dividing one term by a previous term. You can use any two consecutive terms.  1st and 2nd term: | |
| **Step 3:** Write the recursive formula. | Subscript Notation: | Function Notation: |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Given the recursive formula for the arithmetic sequence , find the first three terms of the sequence. |  |
| P 2 | Write out the first five terms of the sequence . Determine if the sequence is arithmetic or geometric, and then find the common ratio or difference. Which option below is accurate?  Option #1: The sequence is arithmetic, and the common difference is 6.  Option #2: The sequence is arithmetic, and the common difference is −2.  Option #3: The sequence is geometric, and the common ratio is −2.  Option #4: The sequence is geometric, and the common ratio is 2.  Option #\_\_\_ is accurate. | 3 |
| P 3 | What is the common difference in the recursively defined arithmetic sequence .  The common difference is \_\_\_\_. | -5 |
| P 4 | Which option is the correct recursive equation for the arithmetic sequence 6, 2, −2, −6, . . .?  Option #1:  Option #2:  Option #3:  Option #4:  Option #\_\_\_\_ is the correct recursive formula for the sequence. | 2 |
| P 5 | Find the 7th value in the arithmetic sequence described by the explicit formula | 26.5 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Use the explicit formula  to find the seventh term of the arithmetic sequence. |  |
| Q 2 | Given the recursive formula for the geometric sequence , find the second term of the sequence. |  |
| Q 3 | Use the recursively defined geometric sequence  and find the common ratio. | 4 |
| Q 4 | Find the recursive formula for the arithmetic sequence 7, 5, 3, 1, −1, . . . |  |
| Q 5 | Which of the following correctly uses the explicit formula, , to find the 9th term of the described arithmetic sequence? |  |

**Lesson 4 – Arithmetic Sequences**

**Key Words:**

* **arithmetic sequence** – an ordered list of terms in which the difference between any two consecutive terms remains constant
* **coordinate grid** – a series of consistent increments where the vertical and horizontal lines intersect
* explicit formula – a formula to determine the nth term of different types of sequences (e.g., )
* **linear function** – an equation of the form , in which the variables appear only in the first degree, are multiplied by constants, and are combined only by addition and subtraction
* **recursive formula** – a formula expressing any term of a sequence as a function of one or more preceding terms
* **slope** – the steepness of a line, found by dividing the change in the *y*-value by the change in the *x*-value
* **term** – is either a single number or variable, or numbers and variables multiplied together; can be used to refer to the individual items in a sequence of values
* **term number** – a natural (counting) number that indicates the position of a term in a sequence, starting with 0 or 1; also referred to as an index
* ***x*-axis** – the horizontal axis in the Cartesian coordinate system
* ***y*-axis** – the vertical axis in the Cartesian coordinate system

**Formulas:**

* Arithmetic Formulas
  + Explicit:
  + Recursive: ,
* Linear Function Equation:
* Slope Formula:

**Objective 1:** In this section, you will graph arithmetic sequences on coordinate grids with the term number on the *x*-axis and the term on the *y*-axis.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* Graphing *arithmetic sequences* on a *coordinate grid* can help to predict the terms in a sequence.
* An *arithmetic sequence* creates a linear graph.
  + The *x*-axis is the term number *n*.
  + The *y*-axis is the value of the term.

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| **Example:** Natasha and Marko plan to plant some roses in their flowerbed. They plan on planting the roses in a trapezoid to accommodate the shape of the flowerbed. The longer side of the bed will have **13 roses** and each row will **decrease by 4 roses**.   1. Model this scenario with a graph. 2. How many roses will they need for the 3rd row? | |
| **Step 1:** Create a table that represents the number of roses in each row. | The Row (*n*) is the input and the number of roses is the output.  The number of roses in each row will decrease by 4. |
| **Step 2:** Create a graph. | *x*-axis: Rows (*n*)  *y*-axis: Number of Roses  Create coordinate pairs from the table and plot each point on the graph. Coordinate pairs: (1,13), (2,9), (3,5), (4,1) |
| **Step 3:** Use the graph to answer the question.  How many roses will they need for the 3rd row? | They will need 5 roses for the 3rd row according to the graph. |

**Objective 2:** In this section, you will show that arithmetic sequence formulas have the same structure as linear functions.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* *Arithmetic sequences* and *linear functions* are related:
  + The **common difference** is the **slope**
  + The **first term** of the sequence is the ***y*-intercept**
* Recall the *recursive* and *explicit* formulas for *arithmetic sequences*.

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| **Arithmetic Sequence Formulas** | |
| **Recursive** | **Explicit** |
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* Recall that when an *arithmetic sequence* is graphed it creates a *linear function*.
  + A *linear function equation* produces a straight line with a constant rate of change.

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| **Linear Function Equation** |
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* How to show that an arithmetic sequence has the same structure as linear functions:
  + Use the formula to list the terms in sequence
  + Organize the terms and term numbers in a table
  + Find the slope and *y*-intercept of the table
    - Recall the slope formula:
    - Recall that *b* is the y-intercept in the formula
  + Graph the sequence and the function on the same coordinate plane to confirm they are the same

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| **Example:** The following explicit formula represents a plant’s growth in centimeters () each week (*n*) at a nursery. Use the explicit formula to show that it has the same structure as a linear function. | |
| **Step 1:** Use the formula to list the terms in sequence. | Plug in 1, 2, 3, 4, and 5 for *n*.   |  |  |  | | --- | --- | --- | | n | Calculation: |  | | 1 |  | 3 | | 2 |  | 3.25 | | 3 |  | 3.50 | | 4 |  | 3.75 | | 5 |  | 4 | |
| **Step 2:** Organize the information on a table. |  |
| **Step 3:** Find the slope. | * Pick any two ordered pairs to plug into the formula. For instance (1,3) and (2,3.25). |
| **Step 4:** Find the *y-*intercept. | * Use the slope-intercept form of a linear function: * Plug in *y*, *x*, and m*.* * Solve for *b*. * Pick any point such as (5,4) to plug in for *x* and *y*. * The slope was found in the previous step: |
| **Step 5:** Write the linear function. | * and * Linear function form: |
| **Step 6:** Plot the numbers from the sequence (table in step 2) on the coordinate grid. Also plot the linear function on the same grid. | * Notice that the linear function crosses the arithmetic sequence points. Therefore, the explicit formula has a linear structure. |

**Objective 3:** In this section, you will write functions to represent arithmetic sequences.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* *Arithmetic sequences* and *linear functions* are related:
  + The **common difference** is the **slope**
  + The **first term** of the sequence is the ***y*-intercept**
* An *arithmetic sequence* can be rewritten as a *linear function* and vice versa.
  + Create a table of the sequence if it is not given.
  + Find the slope from the table using slope formula:
  + Find the y-intercept by using the slope-intercept form:
    - Choose one point and the slope to find the *y*-intercept.
    - Plug in the respective values into
  + Rewrite as a linear function:

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| **Example:** Taneka is helping her grandparents stack hay. She places 25 bales in the bottom row and each row above has 2 bales less than the previous row. The sequence of each row begins as 25, 23, 21, ...  Use the arithmetic sequence to write a linear function. | |
| **Step 1:** Create a table of the sequence. | |  |  | | --- | --- | | **Row (*n*)** | **Number of Hay Bales** | | 1 | 25 | | 2 | 23 | | 3 | 21 | | 4 | 19 | | 5 | 17 | |
| **Step 2:** Use the table to find the slope. | * Pick any two points and use the slope formula: * Let’s use the first two points: (1, 25) and (2,23) * The slope is |
| **Step 3:** Find the *y*-intercept. | * Substitute the *x*, *y*, and *m* to find the *y*-intercept or *b* using the slope-intercept form: * Choose any point for the *x* and *y* values, let’s use the point: (4, 19). |
| **Step 4:** Rewrite as a linear function. |  |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | *Use the table to answer the question.*    Mohammad is training for a marathon. He has started a training program to increase his mileage. When Mohammad started training, he could run 1 mile. His goal is to increase his mileage by 2 miles each week. The table shows his longest run in miles with respect to the training week. Which graph models the sequence accurately?  Option #1: Five points showing an increase of longest run over time are plotted on a coordinate plane. The y-axis is labeled longest run left parenthesis miles right parenthesis. The x-axis is labeled weeks.  Option #2: Five points showing an increase of the number of weeks over the longest run are plotted on a coordinate plane. The x-axis is labeled longest run left parenthesis miles right parenthesis. The y-axis is labeled weeks.  Option #3: Five points showing an increase of longest run over time are plotted on a coordinate plane. The y-axis is labeled longest run left parenthesis miles right parenthesis. The x-axis is labeled weeks.  The graph in Option #\_\_\_\_ models the sequence accurately. | 1 |
| P 2 | Use the image to answer the question.  Vienna is creating a video to explain the arithmetic sequence displayed in the logic puzzle. Which graph should she choose to model the sequence accurately?  Option #1: Four points showing an increase in number of squares with an increase in image number are plotted on a coordinate plane. The y-axis is labeled number of squares. The x-axis is labeled image number.  Option #2: Four points showing an increase in number of squares with an increase in image number are plotted on a coordinate plane. The y-axis is labeled number of squares. The x-axis is labeled image number.  Option #3: Four points showing an increase in number of squares with an increase in image number are plotted on a coordinate plane. The y-axis is labeled number of squares. The x-axis is labeled image number.  The graph in Option #\_\_\_ models the sequence accurately. | 2 |
| P 3 | Use the arithmetic sequence formula  to complete the missing values in the table. | 7; 11 |
| P 4 | Use the image to answer the question.  Write the function of the graphed arithmetic sequence.  The function of the arithmetic sequence is \_\_\_\_\_ |  |
| P 5 | Use the image to answer the question.  Analyze the tile pattern and write a function for the pattern. Use x for the image number and y for the number of tiles in each image.  The function modeled by the pattern is \_\_\_, since the slope is \_\_\_ and the y-intercept is \_\_\_\_. | ; 3; 0 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Andre and Kara are collecting food for the food pantry in their neighborhood. For every person they ask to donate, they receive an average of 13 items. Create a table and graph an arithmetic sequence to model this scenario. What would be the **most** appropriate label for the y-axis? | number of items donated |
| Q 2 | Eric has $27.00 and wants to buy comic books. If each comic book costs $4.00, create a table and graph the arithmetic sequence to display how much money Eric has left after each comic book purchase. Which graph is an accurate representation of the problem? | Four points showing a decrease in money remaining with an increase in number of comic books are plotted on a coordinate plane. The y-axis is labeled money remaining. The x-axis is labeled number of comic books. |
| Q 3 | To show that arithmetic sequence formulas have the same structure as linear functions, choose the linear function equation that has the same structure as the arithmetic sequence . |  |
| Q 4 | Use the table to answer the question.    Use the arithmetic sequence formula  to find the missing values in the table. |  |
| Q 5 | Alex and Jesse are baking holiday muffins. On day one, they baked five muffins, on day two, they baked 11 muffins, and on day three, they baked 17 muffins. What is the slope for this arithmetic sequence? | 6 |

**Lesson 5 – Geometric Sequences**

**Key Words:**

* **coordinate grid** – a series of consistent increments where the vertical and horizontal lines intersect
* **explicit formula** – a formula to determine the nth term of different types of sequences (e.g., )
* **exponential function** – an equation of the form , in which the independent variable appears in the exponent
* **geometric sequence** – an ordered list of terms in which the ratio between any two consecutive terms remains constant
* **term** – is either a single number or variable, or numbers and variables multiplied together; can be used to refer to the individual items in a sequence of values
* **term number** – a natural (counting) number that indicates the position of a term in a sequence, starting with 0 or 1; also referred to as an index
* ***x*-axis** – the horizontal axis in the Cartesian coordinate system
* ***y*-axis** – the vertical axis in the Cartesian coordinate system

**Formulas:**

* Geometric Formulas
  + Explicit:
  + Recursive:
* Exponential Function:
* Exponential Equation:

**Objective 1:** In this section, you will graph geometric sequences on coordinate grids with the term number as the horizontal axis and term as the vertical axis.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* Graphing *geometric sequences* on a *coordinate grid* can help to predict the terms in a sequence.
* A *geometric sequence* creates an exponential graph.
  + The *x*-axis is the term number *n*.
  + The *y*-axis is the value of the term.

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| **Example:** Nikola is running for sophomore class president. He sends a political advertisement message to two of his friends and they forward the message to another two friends after a minute. After another minute, the two friends each forward the advertisement to another two friends.  Create a graph to represent the scenario. | |
| **Step 1:** Create a table with minutes (*n*) on the input side and the number of recipients on the output side. |  |
| **Step 2:** Create a graph with the minutes (*n*) on the *x*-axis and the number of recipients on the *y*-axis. |  |

**Objective 2:** In this section, you will show formulas for geometric sequences have the same structure as exponential function equations.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* The structure of a *geometric sequence* and an *exponential function equation* are very similar.
  + In an exponential function, *a* is the **starting value**, like the *a* in a geometric sequence.
  + The *b* is the **common ratio**, like the *r* in a geometric sequence.

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| **Geometric Sequence** | **Exponential Function Equation** |
| Explicit Formula:  Recursive Formula: | Function:  Equation: |

* Recall how to find the *common ratio* if it is not given in the problem:
  + Divide one term by the preceding term
  + Do this a few times and if the answer is the same each time, then you have found the common ratio.

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| **Example:** Thirty flies are in a barn, and their number will double each week. The equation gives the population after *x* weeks. How many flies will there be after **8 weeks**? | |
| **Step 1:** Identify the starting value and the common ratio. | : Recall that *a* is the start value and *b* is the common ratio. |
| **Step 2:** Evaluate the function after 8 weeks. | The exponent will be the number of weeks you are trying to find a value for.  There will be 7,680 flies in the barn after 8 weeks. |
| You can also represent the scenario as a geometric sequence. |  |

* What if you have a set, but want to know the nth term? An [explicit formula](#Bookmark2) is helpful for finding any term in a sequence.

|  |  |
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| **Example:** Find the explicit formula for the sequence 7, 21, 63, 189, ... Then find the seventh term in the sequence. | |
| **Step 1:** Identify the starting value and the common ratio. | * Starting value: * Common ratio:   + Divide one of the numbers in the sequence by the preceding value, for example: . |
| **Step 2:** Use the values from step 1 to write the explicit formula. |  |
| **Step 3:** Find the value of the seventh term by substituting 7 for *n*. Solve. |  |
| **Step 4:** State the answer. | The seventh term in the sequence is 5103. |

**Objective 3:** In this section, you will write functions to represent geometric sequences in given situations.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* A *function* allows you to find any term in the sequence without having to calculate all of the preceding terms.
* A *geometric sequence* can be represented with *exponential functions* by taking the *explicit formula* for a geometric sequence and making a few substitutions.
  + Recall the explicit formula for a geometric sequence:
  + Replace with . Replace *n* with *x*.
  + Replace with the initial value of the sequence.
  + Replace with the common ratio.

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| Explicit Formula |  | Exponential function |
|  |  |  |

* To graph a geometric sequence, you can use the exponential function.
  + Place the *x*-value as the *x*-coordinate.
  + The *y*-value will be the term in the geometric sequence once you have solved the function.

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| **Example:** A geometric sequence has an initial value of 5 and a common ratio of 3. Write a function to represent the sequence and **first four terms in the sequence.** Graph the sequence on the coordinate plane. | |
| **Step 1:** Identify the initial value and the common ratio. | * The initial value is 5. * The common ratio is 3. |
| **Step 2:** Find the first term. | The first term is the initial value of 5. |
| **Step 3:** Substitute 2 for *x* to find the second term. | The second term is 15. |
| **Step 4:** Substitute 3 for *x* to find the third term. | The third term is 45. |
| **Step 5:** Substitute 4 for *x* to find the third term. | The fourth term is 135. |
| **Step 6:** Create a table for the term numbers and terms. | |  |  | | --- | --- | |  |  | | 1 | 5 | | 2 | 15 | | 3 | 45 | | 4 | 135 | |
| **Step 7:** Create coordinates from the table and graph them on the coordinate plane. | The first four coordinates of this sequence are (1,5), (2,15), (3,45), (4,135). |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | A savings account is opened with $15 and increases monthly by 20%. Complete the graph of the geometric sequence by labeling the points. |  |
| P 2 | Jamie and Eddie are buying an apartment for $320,000. Their loan officer tells them that their principal will decrease by 4.5% every year if they make the minimum mortgage payments. Complete the table to predict the principal for the first 3 years. (Round to the nearest hundredth, if applicable.) | 1. 305,600.00 2. 291,848.00 3. 278,714.84 |
| P 3 | Bacteria quadruples on a daily basis. The population starts with 50 bacteria. Using the general exponential equation , what value would represent ?  The value of  is \_\_\_. | 50 |
| P 4 | A rabbit population doubles every month. If the population of rabbits is 32 at the end of the first month, how many rabbits will there be after eight months?  The initial number of rabbits is 1. The common ratio is 2. The number of rabbits after eight months is 3. | 1. 16 2. 2 3. 4,096 |
| P 5 | What common ratio is being solved for in the exponential function | 1.25 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which of the following correctly graphs the geometric sequence? | Five points are plotted on the first quadrant of a coordinate plane. The x-axis ranges from 0 to 6 in unit increments and the y-axis ranges from 0 to 60 in increments of 10. |
| Q 2 | Use the table to answer the question.    The table shows how a text message went viral. What point should be graphed to represent the third term in the geometric sequence? | (3, 27) |
| Q 3 | Which exponential function can be used to show the geometric sequence 40, 160, 640, . . . ? |  |
| Q 4 | Every year that a car is on the road it loses value. If a car loses 15 percent of its value each year and its beginning value is $24,500, what will the value of the car be after five years? | $10,870.78 |
| Q 5 | Write a function to represent the geometric sequence 6, 18, 54, . . . . |  |

**Lesson 6 – Linear Change**

**Key Words:**

* **arithmetic sequence** – an ordered list of terms in which the difference between any two consecutive terms remains constant
* **common difference** – in an arithmetic sequence, the value of the difference between any two consecutive terms
* **linear change** – a pattern created when the relationship between the independent and dependent variables remains constant
* **linear function** – an equation of the form , in which the variables appear only in the first degree, are multiplied by constants, and are combined only by addition and subtraction
* **slope** – the steepness of a line, found by dividing the change in the y-value by the change in the x-value
* ***y*-intercept** – the y-coordinate of a point where a line, curve, or surface intersects the *y*-axis

**Formulas:**

* Linear Function:
* Linear Equation:
* Slope Formula:
* Arithmetic Sequence
  + Explicit:

**Objective 1:** In this section, you will create sequences for situations that describe linear change.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* Recall that an *arithmetic sequence* has a *common difference*.
* Recall that to find a term in an *arithmetic sequence*, you can add the *common difference* to its previous term.
* An *arithmetic sequence* describes a *linear change* in its terms because the terms change in a constant direction.

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| **Example:** Does the following sequence describe a linear change? | |
| **Step 1:** Identify if there is a common difference between each term. | * second term – first term * third term – second term * fourth term – third term |
| **Step 2:** State the answer. | The terms of the sequence are constantly increasing by 9. Thus, the sequence is an arithmetic sequence and describes a linear change. |

* There are situations in the real world that describe *linear change*. You can create corresponding sequences for these situations.

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| **Example:** A water tank containing 1,400 gallons leaks at a rate of 8 gallons per week. Create a sequence that corresponds to the remaining amount of water in the tank. | |
| **Step 1:** Add the common difference to the previous week's amount to create a sequence. | At 0 weeks, before the leak begins, the amount in the tank is 1,400 gallons. Each week after that, the tank **loses 8 gallons**.   |  |  |  | | --- | --- | --- | | **Weeks** | **Calculation** | **Amount in Tank** | | 0 |  |  | | 1 - (the leak begins) |  | 1392 | | 2 |  | 1384 | | 3 |  | 1376 | | 4 |  | 1368 | |
| **Step 2:** State the answer. | The sequence that corresponds to the remaining amount of water in the tank is:  It is an arithmetic sequence with a common difference of –8. |

**Objective 2:** In this section, you will create formulas and graphs for situations depicting linear change.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* Any situation that describes *linear change* can be represented using a *linear function*.

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| **Linear Function** |
|  |

* The slope (*m*) is the rate of change of the *y*-value over the *x*-value.
  + The *slope* corresponds to the *common difference* between the two units in an *arithmetic sequence*.

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| **Slope Formula** |
| The slope of a line containing points and is solved using the formula: |

* To better understand the relationship that exists between two quantities, you can create formulas and graphs for any situation that describes linear change.

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| **Example:** You want to buy a new TV for your room. You start by saving $100, then save an additional $25 per week. The table shows your progress so far. Model the relationship using a sequence, a function, and a graph.   |  |  | | --- | --- | | **Week Number** | **Total Amount** | | 0 | $100 | | 1 | $125 | | 2 | $150 | | 3 | $175 | |
| **Model with an Arithmetic Sequence** |
| * From the given table, we see that the sequence is 100, 125, 150, 175, ... * Recall the explicit formula of an arithmetic sequence:   + Identify the first term:   + Identify the common difference: * Write the formula: |
| **Model with a Function** |
| * Recall a linear function is represented by: * Identify the slope using any two points given in the table:   + For example: (0,100) and (3,175)   + Apply the slope formula:   + The slope is * Use the table to identify the *y*-intercept. Recall that the y-intercept is the y-value when .   + From the table, when , the *y*-value is 100.   + The y-intercept is * Write the function: |
| **Model with a Graph** |
| * The function can be used to graph the relationship. * Notice that the graph also satisfies the sequence where , , , and .      * Saving a constant amount of money each week represents linear change. Each week your savings increase by $25. |

**Objective 3:** In this section, you will discuss the features of graphs that show linear change.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* *Linear change* is a pattern that is created when the relationship between the dependent and independent variables remains **constant**.
  + Much like in an *arithmetic sequence* where each term changes by a constant or *common difference*.
* You can create a graph to model a linear or arithmetic sequence. This can be done from a table or an equation.
  + Recall the equation of linear functions:
  + To determine which points to plot, you can choose *x*-values to substitute into the equation and solve for *y*-values.
* Two key features of a graph showing *linear change* are the *initial value,* also called the *y*-intercept*,* and the *rate of change*, also called the *slope*.

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| **Example:**  You are working with an arithmetic sequence in which and . Graph the first four points of the sequence. What is the rate of change? | |
| **Step 1:** Determine the first four points by adding the common difference to the *a* values. | Recall that the common difference is .   * Starting with |
| **Step 2:** Write the points as ordered pairs. | (1,6), (2,16), (3,26), (4,36) |
| **Step 3:** Graph the points on the coordinate plane. |  |
| **Step 4:** Find the rate of change. | * The *y*-values increase by 10 for every *x*-value increase of 1. * The rate of change is 10. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Use the image to answer the question.  The tables for a party are seated as 4 people at 1 table, 6 people at 2 tables, and 8 people at 3 tables. Use a sequence to determine how many tables would be needed for 16 people. | 7 |
| P 2 | Use the table to answer the question.  Let *x* represent the time in minutes and let *y* represent the distance in feet.    Use the table to create a formula for the given information. |  |
| P 3 | Use the image to answer the question.  Does the graph accurately depict the formula ? Enter 1 for yes or 2 for no. | 1 |
| P 4 | Use the image to answer the question.  What would be the next point on the graph?  (\_\_\_, \_\_\_) | 12; 2 |
| P 5 | Use the image to answer the question.    What is the rate of change in the graph? | 2 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Using the equation , create a sequence that corresponds from the 1st term to the 5th term. | 20, 33, 46, 59, 72 |
| Q 2 | Create a formula for the points (1,5), (2,7), (3,9), and (4,11). |  |
| Q 3 | Which graph accurately depicts the correct formula for the points (1,5), (2,7), (3,9), and (4,11)? | A line passes through two points plotted on a coordinate plane. |
| Q 4 | When discussing the key features of a linear graph, which of the following statements is true? | The points will make a straight line. |
| Q 5 | Use the image to answer the question.    What would the next point be on the graph? | (5, -2,5) |

**Lesson 7 – Exponential Growth**

**Key Words:**

* **common ratio** – in a geometric sequence, the value of the ratio between any two consecutive terms
* **exponential equation** – a mathematical statement where the exponent is a variable
* **exponential growth** – an increasing pattern in which the increase gets steeper over time
* **geometric sequence** – an ordered list of terms in which the ratio between any two consecutive terms remains constant

**Formulas:**

* Exponential Function:
* Exponential Equation:
* Growth Factor: , where *r* is the rate of growth
* Geometric Sequence
  + Explicit Formula:

**Objective 1:** In this section, you will create sequences for exponential growth situations.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* *Exponential growth* can be modeled using *geometric sequences*.
* Recall that a *geometric sequence* increases or decreases by a *common ratio*.
  + The *common ratio* is represented by the variable *r*.
  + You can determine the value of *r* by dividing any term in the sequence by the preceding term.
* You can create a sequence if you are given the *common ratio* and the *initial value*, also known as the first term.

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| **Example:** Does the following sequence model exponential growth?  3, 9, 27, 81, ... | |
| **Step 1:** Identify if there is a common ratio between the terms. | * second term / first term * third term / second term * fourth term / third term |
| **Step 2:** State the answer. | Yes, the sequence models exponential growth because it is a geometric sequence. There is a common ratio of 3 between each term. |

**Objective 2:** In this section, you will create graphs and equations for exponential growth situations.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* Exponential growth can be modeled using equations and graphs.

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| **Exponential Growth Function** |
| or  When given a percentage of growth, you need to calculate the growth factor.  Where is the rate of growth.  For example, if something grows at 12%, then . This means that the growth factor will be . |

* In exponential growth, the growth factor (*b*) will always be greater than 1 because the function is increasing over time.

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| **Example 1:** A boat accidentally brought 12 zebra mussels into a lake. The population of mussels increases 40% each week. Write an equation to model the growth of the zebra mussel population over time. | |
| **Step 1:** Identify the initial value. | The initial value is 12. |
| **Step 2:** Identify the growth factor. | The population grows at a rate of 40% each week.  Growth Factor:   * First, calculate the rate: * Next, calculate the growth factor: |
| **Step 3:** Write an exponential equation. |  |

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| **Example 2:** Imagine a virus that spreads exponentially, tripling each day. The equation models the number of people who are infected by the virus. Create a graph to show the pattern for the first four days of the spread. | |
| **Step 1:** The equation is given, so you can substitute for *x* to find corresponding *y* values. |  |
| **Step 2:** Create a table of the values you found. | |  |  | | --- | --- | | **Days (*x*)** | **Patients (*y*)** | | 0 | 10 | | 1 | 30 | | 2 | 90 | | 3 | 270 | | 4 | 810 | |
| **Step 3:** Use the table to create a graph of the function. Recall that each *x* and *y* pair is a coordinate point. | The coordinates to graph are (0,10), (1,30), (2,90), (3,270), (4,810). |

**Objective 3:** In this section, you will discuss the key features of graphs that show exponential growth.

*Mathematical Practice Standard: Look for and make use of structure.*

**Big Ideas:**

* A *geometric sequence* models *exponential growth*.
* Recall that the explicit formula for a *geometric sequence* is:
* You can convert geometric sequences into an [exponential equation](#Bookmark3) modeled by
  + *a* is equivalent to , the first term or *initial value*. The *initial value*, when graphed, is also the *y*-intercept.
  + *b* is equivalent to *r*, the common ratio or growth factor.
* Recall that the *growth factor*, *b*, is always equal to the sum of 1 and the percent of growth as a decimal, called the rate.

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| **Example:** Consider the following graph of an exponential growth sequence. What is the initial value and what is the common ratio? | |
| **Step 1:** Identify the initial value. | * Recall that the initial value, when graphed, is also the *y*-intercept. * The line crosses the y-axis at (0,8), which means the initial value is 8. |
| **Step 2:** Find the common ratio. | * To find the common ratio, identify consecutive y-values and divide the latter term by the former. * For example, two consecutive *y*-values in the graph are 8 and 11.2.    + You can perform this calculation on any two consecutive *y*-values and get the same common ratio. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | A chain email tells people to forward the email to 10 friends in order to have good luck. Assuming that everyone in the chain actually follows the directions, create a sequence to show how many people will get the email after 5 rounds of forwarding.  10, 100, 1, 2, 3 . . . | 1. 1,000 2. 10,000 3. 100,000 |
| P 2 | Use the image to answer the question.    A boat washes up onto an island, introducing 25 rabbits into the environment. The rabbits multiply at a rate of 50 percent every month. Does this graph accurately model the situation? Enter 1 for yes or 2 for no. | 2 |
| P 3 | Use the image to answer the question.    Use the graph to fill in the missing elements of the exponential growth equation.  *y* = \_\_\_(\_\_\_\_)*x* | 4; 1.9 |
| P 4 | Use the image to answer the question.  A coordinate plane's x-axis ranges from 0 to 5 by 0.25 unit increments but labeled at every 1-unit intervals. The y-axis ranges from 0 to 10 by increments of 0.5, but labeled at intervals of 2 units. A curve is plotted passing through 4 marked and labeled points.  What is the common ratio of this sequence? | 1.2 |
| P 5 | Use the image to answer the question.  A coordinate plane's x-axis ranges from 0 to 5 in 0.25 unit increments but labeled at every 1-unit interval. The y-axis ranges from 0 to 100 in increments of 5, but labeled at every 20-unit interval. A curve is plotted passing through four marked and labeled points.  What is the initial value of this sequence? | 8 |

**Quick Check Questions and Answers**

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| --- | --- | --- |
|  | Question | Answer |
| Q 1 | Create a sequence to show exponential growth where the population of 120 is doubling every day. | 120, 240, 480 . . . |
| Q 2 | Select the response that creates an accurate formula for the sequence 2, 3.5, 6.125. 10.71875 . . . |  |
| Q 3 | Use the image to answer the question.  Which of the following situations could describe the graph? | Juveldy puts $20 in a high-yield savings account. His investment triples every year. |
| Q 4 | Lori buys five goldfish for her tank. Her research shows that they will reproduce at a rate of 32 percent each month. Which statement discusses a key feature of the graph of this situation? | The points on the graph will have a common ratio of 1.32. |
| Q 5 | Use the image to answer the question.  What is the initial value of the sequence? | 0.5 |

**Lesson 8 – Exponential Decay**

**Key Words:**

* **common ratio** – in a geometric sequence, the value of the ratio between any two consecutive terms
* **exponential decay function** – a mathematical function in which an independent variable appears in one of the exponents, written in the form , where and
* **exponential function** – an equation of the form , in which the independent variable appears in the exponent
* **geometric sequence** – an ordered list of terms in which the ratio between any two consecutive terms remains constant
* **horizontal asymptote** – a horizontal line that the graph of a function approaches as the *x*-value of the function gets very large, approaching infinity, or very small, approaching negative infinity
* **initial amount** – The starting amount or value when in the exponential function, represented by the variable *a* in the equation , where and
* **rate of decay** – The rate at which a quantity is decaying, represented by the variable *r* in the equation , where and
* ***y*-intercept** – the y-coordinate of a point where a line, curve, or surface intersects the y-axis

**Formulas:**

* Geometric Sequence
  + Explicit Formula:
* Exponential Function:
* Exponential Equation:
* Decay Factor: , where *r* is the rate of decay.

**Objective 1:** In this section, you will create sequences that describe exponential decay.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* When a value decays exponentially, *exponential decay*, the *initial value* declines quickly at the beginning and then begins to level out as it approaches zero.
* A *geometric sequence* models exponential decay, just as it models exponential growth.
* In exponential growth, you will see the values increase by a common ratio.
  + The *common ratio* for *exponential growth* will always be greater than 1.
* In *exponential decay*, you will see the values **decrease by a common ratio**.
  + The *common ratio* for *exponential decay* will always be between 0 and 1.

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| **Example:** Write the first five terms in a geometric sequence in which the common ratio, *r*, is , and the initial value, , is 10. | |
| **Step 1:** Calculate each term of the sequence by multiplying the preceding term by the common ratio. | |  |  |  | | --- | --- | --- | | **Term Number** | **Calculation** | **Term** | |  |  | 10 | |  |  | 7.5 | |  |  | 5.625 | |  |  | 4.21875 | |  |  | 3.164062 | |
| **Step 2:** Write the sequence. | The first five terms of the sequence are:  10, 7.5, 5.625, 4.21875, 3.1640625, ... |

**Objective 2:** In this section, you will create formulas and graphs to represent exponential decay problems.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* Recall that, for a function to be exponential, the *y*-values must represent a *geometric sequence*, thus the *common ratio* must all be the same.
* In *exponential decay*, you will see the values **decrease by a common ratio**.
  + The *common ratio* for *exponential decay* will always be between 0 and 1.

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| **Exponential Decay Function** |
| or  When given a percentage of decay, you need to calculate the decay factor.  Where is the rate of decay.  For example, if something decays at 12%, then . This means that the decay factor will be . |

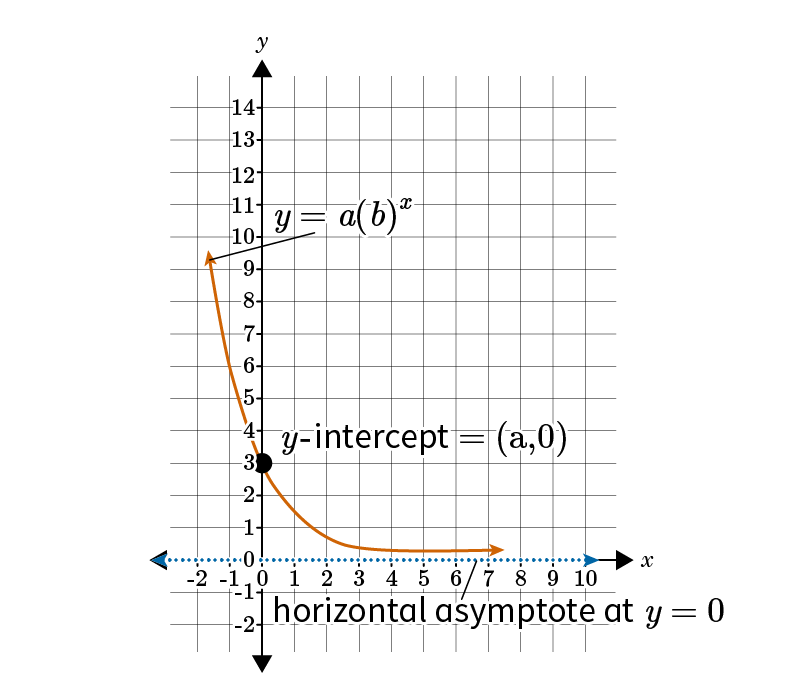
|  |  |
| --- | --- |
| **Example:** Create a function to model the depreciation of the car. The initial price of the car was $30,000, and the car depreciated, or lost value, by 25% each year.   1. Determine how much the car will be worth after 5 years. 2. Graph the function on the coordinate plane. | |
| **Step 1:** Find the initial value. | The initial value of the car was $30,000. |
| **Step 2:** Find the decay factor. | The car loses value by 25% each year.   * First, calculate the rate. * Next, calculate the decay factor. |
| **Step 3:** Substitute the values into the exponential equation. |  |
| **Step 4:** Determine the worth of the car after 5 years by substituting 5 for *x*. | After 5 years, the car will be worth approximately $7,119.14. |
| **Step 5:** Create an input-output table to identify the points for graphing. |  |
| **Step 6:** Plot the points from the table to create a graph. |  |

**Objective 3:** In this section, you will discuss the key features of graphs showing exponential decay.

*Mathematical Practice Standard: Look for and make use of structure.*

**Big Ideas:**

* Recall the [e*xponential decay* function](#Bookmark4).
  + Recall that the *decay factor*, *b*, must be between 0 and 1.
* The *y*-intercept of an *exponential function* in the form will always be at the point , where *a* is the initial value.
  + The *y*-intercept is where the graph crosses the *y*-axis and is also known as the ***initial value***.
* In *exponential decay*, as the *x*-values get larger, the *y*-values get closer to zero.
  + The line is known as the *horizontal asymptote*.



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| **Example:** Given the exponential sequence, which can be modeled by the exponential function and its graph, identify the y-intercept and the horizontal asymptote of the function. | |
| **Step 1:** Identify the *y*-intercept. | * The y-intercept should be at , where is the initial value in the function . * On the graph, the y-intercept is at (0,3). |
| **Step 2:** Identify the horizontal asymptote. | * The horizontal asymptote is the line that the y-value of the graph approaches as the x-value gets larger. * On this graph, the horizontal asymptote is the line . Indicated by the blue dotted line on the graph. |

**Practice Questions and Answers**

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| --- | --- | --- |
|  | Question | Answer |
| P 1 | Drag and drop the expressions into the correct locations.  Perry buys a bag of 20 carrots on the first day. The second day, he eats half of the bag. Each day after, he eats half of what is left in the bag. Create a sequence formula that describes the number of carrots Perry will have on any given day. | 1. 20 |
| P 2 | The equation  represents the decay of a specific species of bird. Select the option below that represents the rate of decay for the species.  Option #1: The rate of decay is 0.05%.  Option #2: The rate of decay is 203.  Option #3: The rate of decay is 5%.  Option #\_\_\_ in the correct rate of decay. | 3 |
| P 3 | In a certain geographic location, a herd of antelopes is declining at a rate of six percent every year. If there are currently 570 antelopes in the herd, create an exponential decay function to model the problem. If the decrease in population continues at this rate, how many antelopes will be in the herd after 10 years? Round your answer to the nearest whole number.  After 10 years there will be \_\_\_\_ antelopes. | 307 |
| P 4 | Use the image to answer the question.  Use the graph of the exponential decay function to determine the equation for the horizontal asymptote of the sequence that is modeled by the graph.  The horizontal asymptote is at *y* = \_\_\_\_\_. | 0 |
| P 5 | Using the discussed properties of exponential decay graphs, find the y-intercept of the function  that models an exponential sequence.  The y-intercept is at the point (0, \_\_\_\_). | 30 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | A local bakery opened with new equipment worth $80,000. Ten years later, the owners’ accountants told them that their equipment had depreciated at a rate of 6 percent per year. Which equation would you use to create a sequence modeling this situation? |  |
| Q 2 | In a certain geographic location, a herd of elephants is declining at a rate of four percent every year. If there are currently 62 elephants in the herd, create an exponential decay function to model the problem. Let y represent the number of elephants after t years. |  |
| Q 3 | Use the table to answer the question.    Complete the input-output table and determine which graph matches the sequence of the decay rate of a specific species of bird modeled by the equation | A coordinate plane's x-axis ranges from 0 to 30 by 5-unit increments and its y-axis ranges from 0 to 6000 by 1000-unit increments. The x-axis is labeled 'Years' and the y-axis is labeled 'Number of Birds.' |
| Q 4 | Use the image to answer the question.    Use the graph of the exponential decay function that models an exponential decay sequence to discuss the properties and determine the equation for the horizontal asymptote of the graph. | The horizontal asymptote is at . |
| Q 5 | Which graph has the following properties?   1. The y-intercept is at (0,22). 2. The horizontal asymptote is at y=5. 3. The graph is decreasing for all values of x. |  |

**Lesson 9 – Linear vs. Exponential Growth Rates**

**Key Words:**

* **arithmetic sequence** – an ordered list of terms in which the difference between any two consecutive terms remains constant
* **exponential function** – an equation of the form , in which the independent variable appears in the exponent
* **geometric sequence** – an ordered list of terms in which the ratio between any two consecutive terms remains constant
* **linear function** – an equation of the form , in which the variables appear only in the first degree, are multiplied by constants, and are combined only by addition and subtraction

**Formulas:**

* Exponential Function:
* Linear Function:

**Objective 1:** In this section, you will prove that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas**:

* Recall that *linear functions* have equations that take the form . When graphed, they look like straight lines.
  + *Linear functions* grow by **equal differences** over equal intervals.
  + Every time the value of *x* increases by 1, the value of *f(x)* increases or decreases by *m*, the slope.
  + If the slope, *m*, is positive, the function will increase.
  + If the slope, *m*, is negative, the function will decrease.
* Recall that *exponential functions* have equations that take the form . When graphed, they increase or decrease very quickly, and then level out.
  + *Exponential functions* grow by **equal factors** over equal intervals.
  + Every time the value of *x* increases by 1, the value of *f(x)* increases or decreases by the growth or decay factor, *b*, or multiplier.
  + If the multiplier, b, is greater than 1, the function will increase.
  + If the multiplier, b, is between 0 and 1, the function will decrease.

**Objective 2:** In this section, you will use sequences to calculate rates of change and show that the growth rate of an exponential function will always eventually exceed the growth rate of a linear function.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* *Arithmetic sequences* are examples of *linear functions* because there is a *common difference* between consecutive terms.
  + Recall that *linear functions* grow by **equal differences** over equal intervals.
  + *Linear functions* have a constant rate of change, which means they grow at the same rate throughout the entire domain of a function.
* *Geometric sequences* are examples of exponential functions because there is a *common ratio* between consecutive terms.
  + Recall that *exponential functions* grow by **equal factors** over equal intervals.
  + *Exponential functions* rate of changewill get progressively greater (for growth) or smaller (for decay) over the domain of the function.

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| **Example:** Consider the following two functions.  Linear function:  Exponential function:   1. Use sequences to calculate the rates of change of each function. 2. Compare the growth rate of each function. | |
| **Step 1:** Model the linear function as an arithmetic sequence. Model the exponential function as a geometric sequence. Place both sequences in the same table. | * Write each function as a sequence by substituting into each function. |
| **Step 2:** Find the rate of change of the linear function. | * Find the rates of change by subtracting the values of consecutive terms. * The sequence for consistently grows by 2.75. |
| **Step 3:** Find the rate of change of the exponential function. | * Find the rates of change by subtracting the values of consecutive terms. * The sequence for does not row at a consistent rate. The rate keeps increasing. |
| **Step 4:** Compare the growth rates of each function. | The growth rate for stays constant, while the growth rate for is always increasing. |

**Objective 3:** In this section, you will use graphs to show that the growth rate of an exponential function will always eventually exceed the growth rate of a linear function.

*Mathematical Practice Standard: Model with mathematics.*

**Big Ideas:**

* The *growth rate* of an *exponential function* will **always eventually surpass** the growth rate of a *linear function*.

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| **Example:** Consider the linear function and the exponential function . Model each function on the same graph and compare the growth rates. | |
| * The growth rate of stays constant while the growth rate of always increases. * Over time, the graph of becomes steeper and steeper. * By , the growth rate of exceeds the growth rate of . | |

* The *exponential function* will often have the lesser growth rate to start but will always have the greater growth rate towards the end. This is true regardless of how big the growth rate is of a linear function.

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| **Example:** Consider the linear function and the exponential function . Model these functions on the same graph and compare the growth rates. | |
| * Around , the growth rate of *g(x)* really starts to increase. * Soon, by , *g(x)* is steeper than *f(x)*, and thus *g(x)* is growing faster than *f(x)*. * When , *g(x)* is greater than *f(x)* and is growing much more quickly. | |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Use the table to answer the question.    What could the table be used to prove?  Option #1: Over equal intervals, linear functions grow by equal differences.  Option #2: Over equal intervals, linear functions grow by equal factors.  Option #3: Over equal intervals, exponential functions grow by equal differences.  Option #4: Over equal intervals, exponential functions grow by equal factors. | 4 |
| P 2 | Use the table to answer the question.    Miguel is planning to invest $5,000. He plans to withdraw his investment and earnings in approximately 10 years. With Option #1, his money will grow according to the function . With Option #2, his money will grow according to the function . He creates sequences for both functions. Which option has the greater rate of change between years 9 and 10? Which option should he choose for his money?  Option #\_\_\_has the greater rate of change between years 9 and 10. If Miguel plans to invest his money for only 10 years, he should choose Option #\_\_\_. | 1; 2 |
| P 3 | Use the image to answer the question.  Consider the graph of Function #1 which is linear and Function #2 which is exponential. How do the growth rates of the two functions compare?  Initially, Function #\_\_\_ has the greater growth rate. By , the growth rate of Function #\_\_\_ surpasses the growth rate of Function #\_\_\_. | 1; 2; 1 |
| P 4 | Use the image to answer the question.  Given that the graph is linear and the graph  is exponential, solve the following problem:  The graph of  and  is shown. Which of the following statements is **true**?  Statement #1: The growth rate of  is greater than the growth rate of between  and .  Statement #2: While the growth rate of  is initially greater than the growth rate of , the growth rate of  keeps increasing and, by , surpasses the growth rate of  .  Statement #3: While the growth rate of is initially greater than the growth rate of  , the growth rate of   keeps increasing and, by , surpasses the growth rate of . | 2 |
| P 5 | Use the image to answer the question.  Given that the graph  is linear and the graph  is exponential, solve the following problem:  Juan is planning to invest $5,000. With Option #1, his money will grow according to the function . With Option #2, his money will grow according to the function . He decides to graph both functions. What does he observe from the graph?  For the first 10 years, Juan’s money will grow faster with Option #\_\_\_, but after that it will grow faster with Option #\_\_\_. | 2; 1 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Use the table to answer the question.    Ramona wants to use the table to prove something about how exponential functions grow. What does she need to do next and what will she be able to prove? | Ramona needs to add another column to the table and find the growth factors between the consecutive values of . She will be able to prove that exponential functions grow by equal factors over equal intervals. |
| Q 2 | Liam is using sequences to compare the growth rates of  and . Which statement correctly describes how Liam should do this and what he will observe? | Liam should compare the rates of change of the terms in both sequences. The growth rate of will quickly surpass the growth rate of . |
| Q 3 | Austin is using graphs to compare the growth rates of  and . Which statement correctly describes how Austin should do this and what he will observe? | Austin should compare the steepness of the curves. The growth rate of will quickly surpass the growth rate of . |
| Q 4 | Use the image to answer the question.  Given that the graph  is linear and the graph  is exponential solve the following problem:  Consider the graph of Function #1 and Function #2. How do the growth rates of the two functions compare? | Initially, Function #2 has the greater growth rate. After , the growth rate of Function #1 surpasses the growth rate of Function #2. |
| Q 5 | Use the image to answer the question.  Marshall is comparing the growth rates of  and  using this graph. Based on the graph, he concludes that the growth rate of  is always greater than the growth rate of . Where is his mistake? | Marshall is only considering a small portion of the domains of the functions. If he went farther to the right on the graph, he would observe that the growth rate of eventually surpasses the growth rate of for large values of *x*. |