Algebra 2 A

**Complex Numbers**

**Unit Summary:** In this unit, you will solve equations that require work with imaginary and complex numbers. You will learn to add, subtract, and multiply with complex numbers. You will also see how the complex solutions to an equation relate to the graph of the equation.

**Lesson 2 – No Real Solutions**

**Key Words:**

* **real number** – any number that can be expressed on a number line that spans from negative infinity to positive infinity
* **solution** – a value that replaced a variable that makes an equation true
* **system** – a collection of equations that contain the same variables

**Formulas:**

* Imaginary Number: $i=\sqrt{-1}$
* Quadratic Formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$

**Objective 1:** In this section, you willcharacterize graphs and equations that have no real solutions.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas**:

* For an equation to have a real *solution*, there must be a *real number* that makes the value of the equation equal to zero.
	+ Recall that *real number*sinclude integers, rational numbers, and irrational numbers.
* For example, the equation $y=3x-3 $has a real solution of $x=1 $that makes the equation equal to zero.
	+ This can be seen on the graph of the equation by looking at the *x*-intercept.
	+ 
* There are many other types of equations that exist and have one, many, or no real solution(s). This lesson will focus on characteristics of equations and graphs that have *no real solution*.
* A graph of an equation has *no real solution* if there is no point of intersection on the graph between the curve itself and the x-axis. In other words, the graph does not cross the *x-*axis.

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| **Example:** Examine the following graph of $y=x^{2}+1$. Is there a point at which $x^{2}+1=0$?  |
|  | * If $y=x^{2}+1$ has any real solution(s), then the graph will intersect the *x-*axis.
* The equation $y=x^{2}+1$ does not intersect the x-axis.
* No value of x will satisfy the equation $x^{2}+1=0$.
* This equation has **no real solution**.
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* You can also determine whether an equation has no real solution by solving the equation.
	+ When you solve the equation for the variable, you will get an answer that is not possible, or not a real number.
	+ Recall that an imaginary unit, $i $, is defined by $i=\sqrt{-1}$. When a solution has a negative number under the square root, this indicates that the solution is not a real number because it is not possible to take a square root of a negative number.

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| **Example:** Decide whether the equation $5x^{2}+3=2x^{2}-5$ has real solutions. |
| **Step 1:** Arrange the equation so that it is set equal to zero.  |  |
| **Step 2:** Solve for the variable (*x*). |  |
| **Step 3:** State the answer. | * $\pm \sqrt{-\frac{8}{3}}$ contains a negative number under the root.
* The equation has no real solutions as the solution found contains the imaginary unit $i $.
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| **Step 4:** Verify by graphing.  | The curve never crosses or touches the *x-*axis. |

**Objective 2:** In this section, you will characterize systems of equations and graphs that have no real solutions.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas:**

* Recall that a value is a *solution* to an equation if it makes the equation true.
* The *solution* for a *system* of equations are the values that make all the equations true in the *system* at the same time.
	+ Recall that a *system* is a collection of equations that share variables.
* Some *systems* have *no real solutions*, meaning there are no real values that make all the equations true simultaneously.
	+ Note that the term “*no real solutions*” means that the system or equation has solutions, but they are not *real numbers*.
* You can use multiple methods to identify if a *system* has *no real solutions.*
	+ Graph the equations separately and identify intersection points. No intersection points of the two graphed equations means that no solution exists.
	+ Set two equations in a system equal and solve for the variable. If an imaginary unit, $i=\sqrt{-1}$, appears, then no real solutions exist.

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| **Example:** Consider the system of equation below. How many real solutions does the system of equations have? |
| **Step 1:** Solve the system algebraically. Start by setting the equations equal to one another.  | $$x^{2}+3=1-x$$ |
| **Step 2:** Rearrange so that the equation is set equal to zero.  | $$x^{2}+3=1-x$$$$x^{2}+x+2=0$$ |
| **Step 3:** The result is a quadratic equation. Use the quadratic formula to solve for *x*.  | $$x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$$ |
| **Step 4:** State the answer. | The results show that no real solutions exist to the system as the imaginary unit, $i $, appears, where $i=\sqrt{-1}$. In other words, the square root in the solution contains a negative number underneath. |
| **Step 5:** Verify by graphing.Graph each equation separately and find intersection points.  | The graphs of the two equations do not intersect at any *x-*values. No real solutions exist for this system of equations. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | *Use the image to answer the question.*An upward open parabola is plotted on a coordinate plane. The x-axis ranges from negative 5 to 5. The y-axis ranges from negative 5 to 10. Both in 5 unit increments.Examine the graph. How many real roots does it have?\_\_\_\_\_ real root(s) | 0 |
| P 2 | *Use the image to answer the question.* An upward open parabola is plotted on a coordinate plane. The x-axis ranges from negative 5 to 10 in unit increments. The y-axis ranges from negative 5 to 10 in increments of 1.Examine the graph. How many real roots does it have? | 2 |
| P 3 | How many real solutions does $2x^{2}-4=-3x^{2}$ have? | 2 |
| P 4 | *Use the image to answer the question.*2 upward open parabolas are plotted on a coordinate plane, 1 made of a solid line and the other made of a dotted line. The x-axis ranges from negative 10 to 10 and the y-axis ranges from 0 to 10.Consider the system of equations graphed on the coordinate plane. How many real solutions does the system of equations have? | 0 |
| P 5 | Consider the system of equations:$$\left\{\begin{array}{c} y = x ^{2}+ 3 \\ y = -x ^{2}+ 1 \end{array}\right.$$How many real solutions are there for this system of equations? | 0 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which of the following descriptions accurately characterizes a graph that has no real roots? | a graph that does not touch the *x*-axis |
| Q 2 | *Use the image to answer the question.*A downward open parabola, made of a solid line is plotted on a coordinate plane. The x-axis ranges from negative 5 to 5 and the y-axis ranges from 0 to negative 10, both in increments of 5.How many real solutions does $f (x) = -2x^{2} - 1$ have, based on the graph? | no real solutions because the graph neither touches nor crosses the *x*-axis |
| Q 3 | What are the real solutions, if any, for $-x ^{2}- 3.5 = -12.5$ ? | {-3, 3} |
| Q 4 | Which of the following best characterizes the number of real solutions for the system of equations $y = -2x + 1$ and y$ = -x - 1 $? | The system of equations has no real solutions |
| Q 5 | Which system of equations has no real solutions? | $y=x^{2}+1$ and $y=-x^{2}-1$ |

**Lesson 3 – The Existence of Imaginary Numbers**

**Key Words:**

* **complex number** – a number consisting of a real and an imaginary component
* **element** – a basic member of a mathematical or logical class or set
* **imaginary number** – a number written as a multiple of the square root of negative 1
* **imaginary unit** – a number represented by the letter $i$ where $i^{2}=-1$ and thus defines $i= √-1$
* **real number** – any number that can be expressed on a number line that spans from negative infinity to positive infinity
* **subset** – a set within another set

**Formulas:**

* Complex Numbers: $z=a+bi $
* Multiplication Property of Radicals:$\sqrt{ab}=\sqrt{a}⋅\sqrt{b}$

**Objective 1:** In this section, you will show that the set of imaginary numbers is a subset of the set of all numbers, separate from the set of real numbers.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas**:

* Recall that *real numbers* are made up of rational and irrational numbers.
* An *imaginary unit* is defined by $i $, where $i=\sqrt{-1}$.
	+ $i $, or multiples of $i $, cannot be expressed as a *real number* because it is not possible to take the square root of a negative number.
* *Imaginary numbers* and *real numbers* are separate subsets of *complex numbers*.



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| **Complex Numbers** |
| Complex numbers are written in the form:$$z=a+bi $$The two terms are separated because they exist in different subsets of complex numbers.* $z $represents a complex number
* $a $ represents the part of the complex number that is real
* $bi $represents the part of the complex number that is imaginary

**Example:** Is the number $9-4i $best described as being a real, complex, or imaginary number?The number $9-4i $is a complex number, where 9 is the real number part and $-4i $is the imaginary part. |

**Objective 2:** In this section, you will re-express numbers containing the square root of negative numbers as complex numbers.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* [Recall](#Bookmark1) that *complex numbers* are in the form $a+bi $, where $a $and $b $are both real numbers, and $i $is the *imaginary unit*.
	+ An *imaginary unit* is $i=\sqrt{-1}$.
* Recall that expressions containing square roots of negative numbers are not real numbers. These can be re-expressed as *complex numbers* in the form $a+bi $using the *Multiplication Property of Radicals*.

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| **Multiplication Property of Radicals** |
| The square root of a product can be written as the product of individual roots of the factors, written symbolically as:$$\sqrt{ab}=\sqrt{a}⋅\sqrt{b}$$ |

* The *Multiplication Property of Radicals* allows you to factor out the square root of –1, replacing it with $i $, and then simplifying the radical.

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| **Example:** Re-express this negative square root, $\sqrt{-36}$, as a complex number in the standard $a+bi $. |
| **Step 1:** Use the Multiplication Property of Radicals to factor out the square root of –1.  | $$\sqrt{ab}=\sqrt{a}⋅\sqrt{b}$$$a=36 $ and $b=-1 $$$\sqrt{-36}=\sqrt{36⋅\left(-1\right)}=\sqrt{36}⋅\sqrt{-1}$$ |
| **Step 2:** Replace the square root of –1 with $i $. | $$\sqrt{36}⋅\sqrt{-1}$$$$\sqrt{36}⋅i$$ |
| **Step 3:** Simplify. | $$\sqrt{36}⋅i=6i$$The square root of 36 is 6. |
| **Step 4:** State the answer. | The $\sqrt{-36}$ re-expressed as a complex number in the form $a+bi $is $6i $. $6i $is a complex number, but it only consists of an imaginary part ($a=0 $), and is therefore purely an imaginary number. |

* When solving quadratic equations, the solutions re-expressed as *complex numbers* will often contain two terms. Use the following steps to re-express expressions with multiple terms.
	+ Recall the Square Root Property.

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| **The Square Root Property** |
| If $x^{2}=k$, then:* Take the square root of both sides of the equation:
	+ $\sqrt{x^{2}}=\sqrt{k}$
* On the left side, the square root and power of two cancel out:
	+ $x=\sqrt{k}$
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| **Example:** Find the solution for the equation $\left(x+5\right)^{2}+3=0$. |
| **Step 1:** Isolate the term that is squared.  | $$\left(x+5\right)^{2}+3=0$$$$\left(x+5\right)^{2}=-3$$ |
| **Step 2:** Use the Square Root Principle. Take the square root of both sides and use $\pm $ in the result. | $$\left(x+5\right)^{2}=-3$$$$\sqrt{\left(x+5\right)^{2}}=\sqrt{-3}$$$$x+5=\pm \sqrt{-3}$$ |
| **Step 3:** Solve for *x*. | $$x+5=\pm \sqrt{-3}$$$$x=-5\pm \sqrt{-3}$$ |
| **Step 4:** Apply the [Multiplication Property of Radicals](#Bookmark2). | $$x=-5\pm \sqrt{-3}$$$$x=-5\pm \sqrt{3⋅\left(-1\right)}$$$$x=-5\pm \sqrt{3}⋅\sqrt{-1}$$$$x=-5\pm i\sqrt{3}$$ |
| **Step 5:** State the answer.  | The solutions to the given equation are $x=-5+i\sqrt{3}$ and $x=-5-i\sqrt{3}$.These solutions are complex numbers in the standard form $a+bi $that contain both real number and imaginary number parts. |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Classify the following numbers as complex, imaginary, or real: Option #1: complex Option #2: imaginary Option #3: real$\sqrt{3}$ is a number as shown in Option #\_\_\_\_\_.$5i$ is a number as shown in Option #\_\_\_\_\_.$3+5i$ is a number as shown in Option #\_\_\_\_\_. | 3; 2; 1 |
| P 2 | Given the complex number $\sqrt{7}-6i$, determine the real part and the imaginary part.\_\_\_\_\_ is the real part of the complex number and \_\_\_\_\_ is the imaginary part. | $\sqrt{7}$; $-6i$ |
| P 3 | Re-express $3\sqrt{32}$ as a complex number with the imaginary unit $i=\sqrt{-1}$. | $$12i\sqrt{2}$$ |
| P 4 | Re-express $\sqrt{-18}$ as a complex number with the imaginary unit $i=\sqrt{-1}$. | $$3i\sqrt{2}$$ |
| P 5 | Rewrite $3+4\sqrt{-45}$ as a complex number in the standard form $a+bi$. | $$3+12i\sqrt{5}$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | What are the types of solutions for *x* in the equation $x^{2} + 4 = 0$ ? | imaginary |
| Q 2 | What type of number is $3+4i$? | complex |
| Q 3 | Which of the following re-expresses the negative square root $\sqrt{-40}$ as a complex number in the standard form $a+bi$? | $$2i\sqrt{10}$$ |
| Q 4 | Which of the following re-expresses the negative square root $-6\sqrt{-20}$ as a complex number in the standard form $a+bi$? | $$-12i\sqrt{5}$$ |
| Q 5 | What is the solution to the equation $2(x-16)^{2}=-8$, re-expressed as a complex number in the standard form $a\pm bi$? | $$16\pm 2i$$ |

**Lesson 4 – The Complexity of Numbers**

**Key Words:**

* **complex number** – a number in the form of $a+bi$ where $a $and $b $are real numbers and $i= √-1$

**Objective 1:** In this section, you will show that every number is a complex number composed of a real part and an imaginary part.

*Mathematical Practice Standard: Look for and express regularity in repeated reasoning.*

**Big Ideas**:

* [Recall](#Bookmark1) complex numbers in the form $a+bi $,where $a $is the real part, and $bi $is the imaginary part.
	+ Recall that $i=\sqrt{-1}$.
* Every number, including real numbers, is part of the set of *complex numbers* and can be written as a *complex number*.
* A *complex number* that only has a real part, and no imaginary part, is a real number.
	+ For example, $15+0i=15 $, because $0i=0 $.
	+ Adding $0i $to any real number allows it to be written as a *complex number*.
* A *complex number* that only has an imaginary part, and no real part, is a purely imaginary number.
	+ For example, $0-3i=-3i $.
	+ Adding 0 to any purely imaginary number allows it to be written as a *complex number*.

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| **Number** | **Expressed as Complex Number (**$a+bi $**)** |
| $$-3 $$ | $$-3+0i $$ |
| $$i\sqrt{7}$$ | $$0+i\sqrt{7}$$ |
| $$π$$ | $$π+0i $$ |
| $$6i $$ | $$0+6i $$ |

**Lesson 6 – Properties of Complex Numbers**

**Key Words:**

* **Associative Property** – a rule stating that changing the parentheses in a multiplication or addition expression does not change the value of that expression
* **Commutative Property** – a rule stating that changing the order in a multiplication or addition expression does not change the value of that expression
* **Distributive Property** – a property of multiplication where each term of a sum inside a pair of parentheses gets multiplied by another term outside the parentheses

**Formulas:**

* Commutative Property:
	+ Addition: $a+b=b+a  $
	+ Multiplication: $ab=ba  $
* Associative Property:
	+ Addition: $a+\left(b+c\right)=\left(a+b\right)+c$
	+ Multiplication: $a⋅\left(b⋅c\right)=\left(a⋅b\right)⋅c$
* Distributive Property: $a\left(b+c\right)=ab+ac$

**Objective 1:** In this section, you will show that the Commutative and Associative Properties hold for the set of complex numbers.

*Mathematical Practice Standard: Look for and make use of structure.*

**Big Ideas**:

* Recall the *Commutative and Associative Properties* used for the multiplication and addition of real numbers. These properties also apply when using complex numbers.

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| **Commutative Property**The order in which algebraic terms are added or multiplied together does not affect the sum of those terms. |
| **Real Numbers** | **Complex Numbers** |
| Addition: $a+b=b+a $Multiplication: $a⋅b=b⋅a $ | Addition: $$\left(a+bi\right)+\left(c+di\right)=\left(c+di\right)+\left(a+bi\right)$$Multiplication: $$\left(a+bi\right)⋅\left(c+di\right)=\left(c+di\right)⋅\left(a+bi\right)$$ |
| **Example:** Consider these two expressions $2i⋅3i $and $3i⋅2i $. Which property makes these expressions equivalent?The Commutative Property of Multiplication because the order of multiplication does not matter.$$2i⋅3i=3i⋅2i $$ |

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| **Associative Property**The way factors are grouped in a problem containing only addition and multiplication has no effect on the product. |
| **Real Numbers** | **Complex Numbers** |
| Addition: $a+\left(b+c\right)=\left(a+b\right)+c$Multiplication: $a⋅\left(b⋅c\right)=\left(a⋅b\right)⋅c$ | Addition: $\left(a+bi\right)+ci=i\left(a+b\right)+ci=i\left(a+b+c\right)$ AND$$ai+\left(bi+ci\right)=ai+i\left(b+c\right)=i\left(a+b+c\right)$$Multiplication: $\left(ai⋅bi\right)⋅ci=\left(a⋅b⋅i⋅i\right)=\left(abi^{2}\right)⋅ci=\left(-ab\right)⋅ci=-abci$AND$$ai⋅\left(bi⋅ci\right)=ai⋅\left(b⋅c⋅i⋅i\right)=ai⋅\left(bci^{2}\right)=ai⋅\left(-bc\right)=-abci$$ |
| **Example:** Which property of complex numbers is being applied in the equality?$$9i⋅4i=9⋅4⋅i⋅i $$The Associative Property of Multiplication because the product of three or more terms is the same regardless of how they are grouped.  |

**Objective 2:** In this section, you will show that the Distributive Property holds for the set of complex numbers.

*Mathematical Practice Standard: Look for and make use of structure.*

**Big Ideas:**

* Recall the *Distributive Property* for real numbers. The *Distributive Property* also holds for complex numbers.

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| **Distributive Property**A property of multiplication where each term of a sum inside a pair of parentheses gets multiplied by another term outside the parentheses.  |
| **Real Numbers** | **Complex Numbers** |
| $$a\left(b+c\right)=ab+ac$$ | $$x\left(y+z\right)=xy+xz$$Where *x, y,* and *z* are complex numbers. $$\left(a+bi\right)\left[\left(c+di\right)+\left(e+fi\right)\right]=\left[\left(a+bi\right)\left(c+di\right)\right]+\left[\left(a+bi\right)\left(e+fi\right)\right]$$ |
| **Example:** Calculate the result of the distribution of the following complex numbers. $$\left(2-3i\right)\left[\left(-4+i\right)+\left(-2-i\right)\right]$$ |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Using the Commutative Property, can you say that these two expressions, $2i∙3i$ and $3i∙2i$, are equivalent or not equivalent? Option #1: equivalent Option #2: not equivalent | 1 |
| P 2 | Which property of complex numbers is being applied in the equality?$$9i∙4i=9∙4∙i∙i$$Property #1: the Commutative Property Property #2: the Distributive Property Property #3: the Associative Property | 1 |
| P 3 | Which property of complex numbers is being applied in the equality? $$(4 + 2i) + [(2 + 9i) + (1 + 6i)] = [(4 + 2i) + (2 + 9i)] + (1 + 6i)$$Property #1: the Distributive Property Property #2: the Associative PropertyProperty #3: the Commutative Property  | 2 |
| P 4 | Distribute to simplify the expression $(3i+4)(-2i-5)$.\_\_\_\_\_ + \_\_\_\_\_*i* | -14; -23 |
| P 5 | Distribute to simplify the expression $\left(-3i-5\right)[\left(i+2\right)+\left(2i-4\right)]$.\_\_\_\_\_ + \_\_\_\_\_*i* | 19; -9 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which of the following shows an expression equivalent to the product of two complex numbers, $5i∙4i$? | $$4i∙5i$$ |
| Q 2 | Which of the following correctly simplifies the expression $-17+5+3i-8$ and expresses it in the form of $a+bi$? | $$-20+3i$$ |
| Q 3 | Which of the following is equivalent to the sum of $5i + 3i$? | $$3i + 5i$$ |
| Q 4 | Which of the following correctly shows the application of the Distributive Property to the expression $(7i - 8)(-2i - 5)$ ? | $$54-19i$$ |
| Q 5 | Which of the following is equivalent to $-2i(3i + 4i)$ ? | 14 |

**Lesson 7 – Operations with Complex Numbers**

**Key Words:**

* **Associative Property** – a rule stating that the way factors are grouped in a problem containing only addition or only multiplication has no effect on the product
* **Commutative Property** – a property of algebra that states that the order in which algebraic terms are added together does not affect the sum of those terms
* **Complex conjugate** – if $a+bi$ is a complex number, then $a-bi$ is its conjugate
* **conjugate binomials**– two binomials that have the same terms and only differ by the sign in front of the second term
* **Distributive Property** – a property of multiplication where each term of a sum inside a pair of parentheses gets multiplied by another term outside the parentheses

**Formulas:**

* [Associative Property, Commutative Property, Distributive Property](#Bookmark6)
* Imaginary Numbers: $i^{2}=-1$ and $i=\sqrt{-1}$

**Objective 1:** In this section, you will use the properties of complex numbers to add complex numbers.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas**:

* [Recall](#Bookmark3) that complex numbers are subject to the *Commutative*, *Associative*, and *Distributive Properties*.
* Understanding these properties will help to add and to simplify complex numbers.
* Recall that complex numbers in the form $a+bi $has a real part, $a $, and an imaginary part $b $.
* When adding complex numbers, the real parts are added to each other and the imaginary parts are added to each other.

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| **Example:** Simplify $\left(7+2i\right)+\left(5+6i\right)$. |
| **Step 1:** Rearrange using the Commutative Property.  |  |
| **Step 2:** Regroup the expression using the Associative Property.  |  |
| **Step 3:** Simplify. |  |

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| **Example:** Simplify the expression $6i\left(3+2i\right)$. |
| **Step 1:** Use the Distributive Property to distribute the $6i $to both terms of the complex number.  | $$6i\left(3+2i\right)=$$$$6i⋅3+6i⋅2i $$ |
| **Step 2:** Use the Commutative Property.  | $6i⋅3+6i⋅2i $**=**$$18i+12i^{2}$$ |
| **Step 3:** Simplify. (Remember that $i^{2}=-1$) | $$18+12i^{2}=$$$$18i-12 $$ |

**Objective 2:** In this section, you will use the properties of complex numbers to subtract complex numbers.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* Recall that subtraction of real numbers can be written as addition by adding the opposite of the number you are subtracting.
* Follow these steps when subtracting complex numbers:

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| **Example:** Simplify $\left(3+2i\right)-\left(7-3i\right)$. |
| **Step 1:** Distribute the subtraction to the second term.  | $$\left(3+2i\right)-\left(7-3i\right)=$$$$3+2i-7+3i $$ |
| **Step 2:** Rearrange using the Commutative Property. | $$3+2i-7+3i = $$$$3-7+2i+3i  $$ |
| **Step 3:** Regroup the expression using the Associative Property. | $$3-7+2i+3i = $$$$\left(3-7\right)+\left(2i+3i\right)$$ |
| **Step 4:** Simplify. | $$\left(3-7\right)+\left(2i+3i\right)=$$$$-4+5i $$ |

**Objective 3:** In this section, you will use the properties of complex numbers to multiply complex numbers.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

 **Big Ideas:**

* [Recall](#Bookmark3) the *Commutative, Associative, and Distributive Properties* that can be used when multiplying and simplifying complex numbers.
* When multiplying imaginary numbers, it is important to remember that $i=\sqrt{-1}$ and that $i^{2}=-1$.
* Multiplying complex numbers is like multiplying binomials.
	+ The main difference is that you must be careful when multiplying the imaginary numbers together.
	+ When multiplying imaginary numbers, it is important to remember that $i=\sqrt{-1}$ and that $i^{2}=-1$.

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| **Example:** Multiply $\left(-1+3i\right)\left(5-4i\right)$. |
| **Step 1:** Use the Distributive Property. | $$\left(-1+3i\right)\left(5-4i\right)=$$$$-1\left(5-4i\right)+3i\left(5-4i\right)=$$$$-5+4i+15i-12i^{2}$$ |
| **Step 2:** Combine common terms.  | $$-5+4i+15i-12i^{2}=$$$$-5+19i-12i^{2}$$ |
| **Step 3:** Substitute for $i^{2}=-1$. | $$-5+19i-12i^{2}=$$$$-5+19i-12\left(-1\right)=$$$$-5+19i+12 $$ |
| **Step 4:** Rearrange using the Commutative Property so that real parts are together. | $$-5+19i+12= $$$$-5+12+19i $$ |
| **Step 5:** Simplify. | $$-5+12+19i= $$$$7+19i $$ |

* When you multiply two complex numbers, sometimes you get a product that is comprised of both components, real and imaginary. Other times, you get a product that is comprised of only real components.
* When you multiply *complex conjugates* together the result is always a real number with no imaginary components to it.
	+ The *complex conjugate* of a complex number is the same complex number with the sign flipped.
	+ If $a+bi $is a complex number, then $a-bi $is its *complex conjugate*.

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Simplify $(-8-7i)+(7+6i).$ | $$-1-i$$ |
| P 2 | Evaluate $2i-18i$. Provide your answer in the form of $a + bi$. | $$-16i$$ |
| P 3 | Subtract $(3.4+\sqrt{-25)}-(-9-\sqrt{-9})$. Provide your answer in the form of $a + bi$. | $$12.4+8i$$ |
| P 4 | Simplify $6i∙\left(-4\right)i$. | 24 |
| P 5 | Multiply $(-11+8i)(-1-8i)$. | $$75+80i$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Use the properties of complex numbers to simplify completely $\left(7-\sqrt{-64}\right)+6+3i)$. | $$13-5i$$ |
| Q 2 | Use the properties of complex numbers to subtract $-5i - 16i$ . | $$-21i$$ |
| Q 3 | Simplify the expression $\sqrt{-49}-\sqrt{-64}$. | $$-i$$ |
| Q 4 | Use the Distributive Property to simplify $(6 - 3i)(3 - 5i)$ . | $$3-39i$$ |
| Q 5 | Which answer could be the result of two complex conjugates being multiplied together? | 20 |

**Lesson 8 – Complex Numbers & Quadratic Equations**

**Key Words:**

* **complex number** – a number consisting of a real and an imaginary component
* **imaginary number** – a number written as a multiple of the square root of negative 1
* **multiplicity** – the number of times a given factor appears in the factored form of the equation of a polynomial
* **quadratic equation** – any equation of the form $ax^{2}+bx+c=0 $containing one term in which the unknown is squared and no term in which it is raised to a higher power
* **root** – the value of x that makes an algebraic expression equal to zero
* **zeros of a parabola** – the points where a parabola intersects the x-axis; the x-values that make the quadratic equation equal to zero

**Formulas:**

* Quadratic Equation: $ax^{2}+bx+c=0 $

**Objective 1:** In this section, you will test quadratic equations to see that they have two solutions, though the solutions may involve imaginary or complex numbers.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas**:

* *Quadratic equations* always have two solutions.
	+ Two real solutions: when you can solve the equation by factoring using real coefficients
	+ Two imaginary or complex solutions: when you must take the square root of a negative number to solve the equation
* Recall, that when you want to check if a solution makes an equation true, simply substitute the solution into the original equation to check for a true statement.

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| **Example:** Determine which of the potential solutions make the equation $x^{2}+49=0$ true.$x=-7i $ $x=7 $ $x=7i $ $x=i\sqrt{7}$ |
| **Step 1:** Substitute each value into the original equation to check if it creates a true statement.Start with $x=-7i  $. |  |
| **Step 2:** Substitute $x=7 $. |  |
| **Step 3:** Substitute $x=7i $. |  |
| **Step 4:** Substitute $x=i\sqrt{7}$. |  |

**Objective 2:** In this section, you will estimate what the graph of a quadratic equation looks like based on the equation and its roots.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* You can make a rough guess about what the graph of a *quadratic equation* looks like if you have the equation itself and its two solutions.
* You can identify an up turning or down turning parabola from the first term of the equation $ax^{2}+bx+c=0$, the term with the squared exponent.
	+ If *a* is positive, the curve of the parabola will open upward.
	+ If *a* is negative, the curve of the parabola will open downward.
* The *roots,* or solutions*,* of a *quadratic equation* are the values of *x* which makes the equation equal to zero. These can also be used to predict other features of a parabola.
	+ If a quadratic has two real solutions, the parabola will pass through the *x-*axis at the given *roots*, called the *zeros of a parabola*.
	+ If a quadratic has a repeated solution, two of the same solutions, the graph will touch the *x*-axis at a single point.
* If the roots are *imaginary or complex numbers*, then the parabola does not cross the *x-*axis.
	+ If the leading coefficient, $a $, of a *quadratic equation* is **positive (opens up)**: the parabola will be entirely **above** the *x*-axis, including its minimum point.
	+ If the leading coefficient, $a $, of a *quadratic equation* is **negative (opens down)**: the parabola will be entirely **below** the *x-*axis, including at its maximum point.

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| **Examples** | **Graph** |
| $$y=x^{2}-4x+3$$Roots: $x=1 $and $x=3 $* The leading coefficient is positive – the graph will open upward
* The roots are real numbers – the graph will pass through the *x-*axis at the roots
 |  |
| $$y=-x^{2}-8x-16$$Roots: $x=-4 $* The leading coefficient is negative – the graph will open downward
* The root is repeated, or the two roots are the same – the graph will touch the x-axis at this point
 |  |
| $$y=x^{2}-8x+20$$Roots: $x=4+3i $ and $x=4-3i $* The leading coefficient is positive – the graph will open upward
* The roots are complex numbers – the graph will not touch the x-axis, but because it opens upward, it will be entirely above the *x*-axis, including its minimum point
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**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Students solved the equation $x^{2}=6x-12$ and their solutions are shown. Which student found the correct solutions?Student 1: $x=-3\pm \sqrt{21}$Student 2: $x=3\pm i\sqrt{3}$Student 3: $x=3\pm \sqrt{21}$ | 2 |
| P 2 | Determine which of these potential solutions make the equation $x^{2}+2x+3=0$ true?Option #1: $x=-1\pm i\sqrt{2}$Option #2: $x=-1\pm 2i$Option #3: $x=1\pm i\sqrt{2}$ | 1 |
| P 3 | Students solved the equation $x^{2}+4x+8=0$ and their solutions are shown. Which student found the correct solutions?Student 1: $2\pm 2i$Student 2: -4$\pm 4i$Student 3: $-2\pm 2i$ | 3 |
| P 4 | Which of the following statements accurately describes what the graph of $y=4x^{2}+11x-20$ looks like, with solutions of $x=\frac{5}{4}$ and $x=-4$?Statement #1: The graph opens downward. Statement #2: The graph has no zeros. Statement #3: The vertex is to the left of the y-axis. | 3 |
| P 5 | Which of the following statements accurately describes what the graph of $y=x^{2}+4$ looks like, with roots of $x=2i$ and $x=-2i$?Statement #1: The graph opens downward. Statement #2: The graph does not touch the *x*-axis. Statement #3: The graph touches the x-axis in exactly one location. | 2 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Which is the quadratic equation that has the roots $3+i and 3-i$? | $$x^{2}-6x+10=0$$ |
| Q 2 | Which of the potential solutions make the equation $x^{2}+9=0$ true? | $$\{3i, -3i\}$$ |
| Q 3 | Given the equation $y=-10x^{2}+20x+80$ with solutions of $x=-2$ and $x=4$, which of the following identifies the general shape of its associated graph? | The graph opens downward. |
| Q 4 | Given the equation $y=x^{2}-16x+89$ with zeros of $x=8-5i$ and $x=8+5i$, which of the following identifies the general shape of its associated graph? | The graph lies above the *x*-axis. |
| Q 5 | Given the equation $y=3x^{2}-24x+48$ with solutions of $x=4$ and $x=4$, which of the following identifies the general shape of its associated graph? | The graph opens upward. |

**Lesson 9 – Two Solutions for all Quadratic Equations**

**Key Words:**

* **complex number** – a number consisting of a real and an imaginary component
* **discriminant** – the portion of the quadratic formula that determines whether a quadratic equation has real solutions
* **imaginary number** – a number written as a multiple of the square root of negative 1
* **quadratic equation** – any equation of the form $ax^{2}+bx+c=0 $containing one term in which the unknown is squared and no term in which it is raised to a higher power

**Formulas:**

* Quadratic Equation: $ax^{2}+bx+c=0$
* Quadratic Formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$
* Discriminant: $b^{2}-4ac$

**Objective 1:** In this section, you will use the discriminant to determine the number of real solutions to a quadratic equation.

*Mathematical Practice Standard: Use appropriate tools strategically.*

**Big Ideas**:

* To identify if a *quadratic equation* has real solutions without solving, use the *discriminant*.
* Recall the quadratic formula used to find the solutions to a *quadratic equation*.
	+ $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$
* The *discriminant* is the part of the quadratic formula that lies under the radical sign.
	+ Discriminant: $b^{2}-4ac$
	+ To use the *discriminant*, substitute the coefficients of the a, b, and c terms of the quadratic equation $ax^{2}+bx+c=0$ and evaluate.
* You will know how many solutions the equation has depending on if the result is positive, negative, or zero.
	+ Positive: two distinct real solutions
	+ Negative: no real solutions (imaginary or complex solutions)
	+ Zero: real repeated solutions

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| **Example:** Use the discriminant to find the number of real solutions to the equation $x^{2}-12x+61=0$. |
| **Step 1:** Identify the values of *a*, *b*, and *c* from the quadratic equation. | $$ax^{2}+bx+c=0$$$$x^{2}-12x+61=0$$$a=1 $, $b=-12 $, and $c=61 $ |
| **Step 2:** Substitute these values into the discriminant.  | $$b^{2}-4ac$$$$\left(-12\right)^{2}-4\left(1\right)\left(61\right)=-100$$ |
| **Step 3:** State the answer.  | Since –100 is a negative number, the equation will have no real solutions.  |

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| **Example:** Use the discriminant to find the number of real solutions to the equation $x^{2}-3x-28=0$. |
| **Step 1:** Identify the values of *a*, *b*, and *c* from the quadratic equation. | $$ax^{2}+bx+c=0$$$$x^{2}-3x-28=0$$$a=1 $**,** $b=-3 $, and$c=-28 $ |
| **Step 2:** Substitute these values into the discriminant.  | $$b^{2}-4ac$$$$\left(-3\right)^{2}-4\left(1\right)\left(-28\right)=121$$ |
| **Step 3:** State the answer.  | Since 121 is a positive number, the equation will have two distinct real solutions.  |

**Objective 2:** In this section, you will solve quadratic equations and express their solutions as complex numbers.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* When the *discriminant* is a negative number, the quadratic equation has no real solution. However, there are still *imaginary or complex solutions* that you can find.
	+ Discriminant: $b^{2}-4ac$
* If a *quadratic equation* has no real solutions, then factoring is not the best method, it’s best to use the *quadratic formula* or completing the square.
* Recall that you can use the *quadratic formula* to identify the solutions of a *quadratic equation*.
	+ Quadratic Formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$

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| **Example:** Solve for x in the quadratic equation $x^{2}+2x+5=0.$ |
| **Step 1:** Calculate the discriminant.  | $$ax^{2}+bx+c=0$$$$x^{2}+2x+5=0$$$$a=1, b=2, c=5  $$$$b^{2}-4ac$$$$\left(2\right)^{2}-4\left(1\right)\left(5\right)=-16$$The discriminant is negative, so the equation has no real solutions.  |
| **Step 2:** Use the quadratic formula to solve for *x*.  | $$x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$$$$a=1, b=2, c=5  $$$$x=\frac{-2\pm \sqrt{2^{2}-4\left(1\right)\left(5\right)}}{2\left(1\right)}=\frac{-2\pm \sqrt{-16}}{2}=-1\pm 2i$$ |
| **Step 3:** State the answer.  | The two solutions to the quadratic equation are:$x=-1+2i $ and $x=-1-2i $ |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | Use the discriminant to determine the number of real solutions of $x^{2}+14x+24=0$. | 2 |
| P 2 | Use the discriminant to determine the number of real solutions of $x^{2}+10x=-21$. | 2 |
| P 3 | Use the quadratic formula to solve the equation $2x^{2}-9x+11=0$. Express its solutions in the form $a\pm bi$. | $$\frac{9}{4}\pm \frac{\sqrt{7}}{4}i$$ |
| P 4 | Solve the quadratic equation $x^{2}+16=0$. Express its solutions in the form $a\pm bi$. | $$0\pm 4i$$ |
| P 5 | Solve the quadratic equation $3x^{2}+3x+12=10$. Express its solutions in the form $a\pm bi$. | $$-\frac{1}{2}\pm \frac{\sqrt{15}}{6}i$$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Use the discriminant to determine the number of real solutions of $-6x^{2}-108=0$. Which of the following statements gives the correct explanation? | There will be no real solutions since the discriminant is negative. |
| Q 2 | Use the discriminant to determine the number of real solutions of $4x^{2}+3x-7=0$. Which of the following statements gives the correct explanation? | There will be two real solutions since the discriminant is positive. |
| Q 3 | Solve the quadratic equation 6$x^{2}-3x+6=0$. Which of the following expresses its solutions in the form $a\pm bi?$ | $$\frac{1}{4}\pm \frac{\sqrt{15}}{4}i$$ |
| Q 4 | Solve the quadratic equation 2$x^{2}+4x=-12$. Which of the following expresses its solutions in the form $a\pm bi?$ | $$-1\pm i\sqrt{5}$$ |
| Q 5 | Solve the quadratic equation $10x^{2}-2x+13=3$. Which of the following expresses its solutions in the form $a\pm bi?$ | $$\frac{1}{10}\pm \frac{3\sqrt{11}}{10}i$$ |

**Lesson 10 – Complex Numbers & Higher Order Polynomials**

**Key Words:**

* **complex conjugate** – if $a+bi$ is a complex number, then $a-bi$ is its conjugate
* **complex number** – a number in the form of $a+bi$ where $a $and $b $are real numbers and $i= √-1$
* **imaginary number** – a number written as a multiple of the square root of negative 1
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **quadratic equation** –any equation of the form $ax^{2}+bx+c=0 $containing one term in which the unknown is squared and no term in which it is raised to a higher power
* **real number** – any number that can be expressed on a number line that spans from negative infinity to positive infinity
* **x-intercept** – the coordinates (x,0) where the graph touches the x-axis
* **Zero Product Property** – a rule stating that if the product of two or more factors is equal to zero, then at least one of the factors has to be zero

**Formulas:**

* Quadratic Equation: $ax^{2}+bx+c=0 $
* Quadratic Formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$
* Discriminant: $b^{2}-4ac$

**Objective 1:** In this section, you will test solutions to polynomial equations to show they can have real solutions, complex solutions, or both.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas**:

* The degree of the *polynomial* indicates how many solutions there are for the *polynomial* counting multiplicity.
	+ Multiplicity - the number of times a given factor appears in the factored form of the equation of a polynomial
	+ For example, the polynomial $3x^{4}$ will have four solutions counting multiplicity.
* Recall a *quadratic equation*: $ax^{2}+bx+c=0 $
	+ A *quadratic equation* is a *polynomial* with a degree of **two**.
	+ Recall that *quadratic equations* have **two** solutions: two real solutions OR two non-real solutions.
* *Polynomial* equations of a higher order (above two degrees) can have both real and non-real solutions. The solutions to one *polynomial* can be a mix of real and non-real, unlike a quadratic equation.
* To test solutions of polynomials, recall tools from previous lessons like factoring, long division, quadratic formula, and completing the square.

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| **Example:** Given the equation $x^{3}-10x^{2}+34x-40=0$, test the following solution set to prove a polynomial equation can have both real and non-real solutions. $$x=4,  x=3+i,  x=3-i $$ |
| **Step 1:** Factor the polynomial until you are left with linear and quadratic factors.  | Since 4 is a solution, that means a factor of the polynomial should be ($x-4 $). Test it by dividing the polynomial equation by the factor $x-4 $.The result is a quadratic equation $x^{2}-6x+10$, which means that $\left(x-4\right)$ and $x^{2}-6x+10$ are two factors of the original polynomial. $$x^{3}-10x^{2}+34x-40=\left(x-4\right)\left(x^{2}-6x+10\right)$$Yes, $x=4 $is one solution of the polynomial. |
| **Step 2:** Use the quadratic formula on the quadratic equation factor.  | $$x^{2}-6x+10$$$$a=1, b=-6, c=10 $$$$x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}=\frac{-\left(-6\right)\pm \sqrt{\left(-6\right)^{2}-4\left(1\right)\left(10\right)}}{2\left(1\right)}$$$$=\frac{6\pm \sqrt{-4}}{2}=\frac{6\pm 2i}{2}=3\pm i$$The solutions to $x^{2}-6x+10$ are $x=3+i $ and $x=3-i $. |
| **Step 3:** State the answer.  | The full solution set is $x=4, x=3+i $ and $x=3-i $. |

**Objective 2:** In this section, you will show that if a complex number is a solution to a polynomial equation, its conjugate is also a solution.

*Mathematical Practice Standard: Construct viable arguments and critique the reasoning of others.*

**Big Ideas:**

* If a complex number is a solution to a polynomial, its conjugate is also a solution.
	+ Recall that if $a+bi $is a complex number, then $a-bi $is its *complex conjugate*.
* If you can take a polynomial equation that has complex number solutions and transform it into a quadratic formula problem, you can show that the conjugate of the complex number is also a solution. ([like this example](#Bookmark5))
* Recall that the degree of a polynomial indicates how many solutions it has, and **every complex solution has a conjugate that is also a solution**.

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| **Example:** Without solving, describe the possible number of real and non-real solutions for the polynomial equation $4x^{3}+5x^{2}-6x-2$. |
| * The polynomial is of degree three and therefore must have three solutions.
* Possibility #1: there are three real solutions
* Possibility #2: there is one real solution and 2 non-real solutions (a complex number and its conjugate)
 |

**Objective 3:** In this section, you will show that for every polynomial equation, real solutions correspond to *x*-intercepts, but complex solutions do not.

*Mathematical Practice Standard: Construct viable arguments and critique the reasoning of others.*

 **Big Ideas:**

* Real solutions of polynomial equations correspond to *x-intercepts* when graphed.
	+ For example, if $x=2 $, is a solution to a polynomial, when graphed it will have an *x-intercept* at (2,0).
* Non-real solutions (imaginary numbers) **do not** appear on the graph of a polynomial.

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| **Example #1:** Given the following graph, how many real solutions does the polynomial of degree three have? |
| * Recall that real solutions correspond to x-intercepts.
* The polynomial described is of degree three, meaning there are three solutions.
* The graph shows three x-intercepts, so these must be all three real solutions.
* Solutions: $x=-2, x=2, x=0 $
 |
| **Example #2:** Using the graph of $y=-4x^{3}+2x+2$, how many real solutions does the polynomial have? |
| * Recall that real solutions correspond to x-intercepts.
* The polynomial described is of degree three, meaning there are three solutions.
* The graph shows ONE x-intercept at $\left(1,0\right)$, so one solution to is $x=1 $.
* There are no other x-intercepts so the remaining two solutions must be a non-real number and its conjugate.
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**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | One of the solutions to the equation $x^{3}-5x^{2}+6x=0$ is 2. Find the full solution set. Which of the following options correctly describes the solution set? Option #1: It has one real solution and two non-real solutions. Option #2: It has one non-real solution and two real solutions. Option #3: It has three real solutions. | 3 |
| P 2 | The equation $x^{4}+x^{3}-3x^{2}+9x-108=0$ can be factored as $\left(x^{2}+x-12\right)\left(x^{2}+9\right)=0$. Find the full solution set. Which of the following options correctly describes the solution set? Option #1: It has one real solution and three non-real solutions. Option #2: It has two real solutions and two non-real solutions. Option #3: It has four real solutions. Option #4: It has four non-real solutions. | 2 |
| P 3 | Solve the equation $x^{2}-5x+10=0$. Write your conjugate pair solution(s) as two values separated by a $\pm $ sign. | $$\frac{5}{2}\pm \frac{i\sqrt{15}}{2}$$ |
| P 4 | *Use the image to answer the question.*A curve is graphed on a coordinate plane. The plane’s x and y axes range from negative 10 to 10 in unit increments, but labeled very 5 units.How many real solutions does the equation $y=-x^{3}-x^{2}+5x+2$ have? | 3 |
| P 5 | *Use the image to answer the question.*A downward opening parabola is graphed on a coordinate plane. The plane’s x-axis ranges from negative 4 to 4 in unit increments but labeled in increments of 2. The y-axis ranges from negative 4 to 0 in unit increments but labeled every 2 units.How many solutions exist for the quadratic polynomial in the graph?\_\_\_\_\_ real solutions and \_\_\_\_\_ non-real solutions | 0; 2 |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | One of the solutions to the equation $x^{3}+x^{2}-16x-16=0$ is -4. Test the solution to find the full solution set. Which of the following statements correctly describes the solution set? | The equation has all real solutions |
| Q 2 | Which answer shows the conjugate pair solutions to the equation $x^{2}-3x+8=-2$? | $$\frac{3}{2}\pm \frac{i\sqrt{31}}{2}$$ |
| Q 3 | Djamila is looking for conjugate pair solutions to the equation $3x^{2}-5x=-6$. Which of the following options should she choose? | $$\frac{5}{6}\pm \frac{i\sqrt{47}}{6}$$ |
| Q 4 | *Use the image to answer the question.*A curve is graphed on a coordinate plane. The plane’s x and y axes range from negative 4 to 4 in unit increments, but labeled every 2 units.How many real solutions correspond to *x*-intercepts based on the graph? | 1 |
| Q 5 | *Use the image to answer the question.*A curve is graphed on a coordinate plane. The plane’s x and y axes range from negative 2 to 2 in unit increments.How many real and non-real solutions does the graphed equation have? | three real and zero non-real |

**Lesson 11 – The Fundamental Theorem of Algebra**

**Key Words:**

* **complex number** – a number in the form of $a+bi$ where $a $and $b $are real numbers and $i= √-1$
* **degree** – the value of the highest exponent on any variable in a polynomial
* **Fundamental Theorem of Algebra** – a theorem stating that every polynomial equation with one variable has as many roots as the highest exponent among its terms
* **polynomial** – an expression made of one or more algebraic terms, each of which consists of a constant multiplied by one or more variables raised to a nonnegative power
* **repeated solution** – a value that occurs more than once in the solution set of a polynomial equation of degree $n>1$
* **root** – the value of x that makes an algebraic expression equal to zero
* **solution** – a value that replaced a variable that makes an equation true
* **Zero Product Property** – a rule stating that if the product of two or more factors is equal to zero, then at least one of the factors has to be zero

**Formulas:**

* Quadratic Equation: $ax^{2}+bx+c=0$
* Quadratic Formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$

**Objective 1:** In this section, you will examine quadratic equations to learn how the Fundamental Theorem of Algebra applies to quadratic polynomials.

*Mathematical Practice Standard: Construct viable arguments and critique the reasoning of others.*

**Big Ideas**:

* The *Fundamental Theorem of Algebra* states that every polynomial equation with one variable has as many solutions (including repeated solutions) as the highest exponent of one of its terms.
	+ *Repeated solutions* are when the same value occurs more than once in a solution set.
* Quadratic equations, which are of degree two, will have two solutions:
	+ two real solutions (rational or irrational)
	+ two identical solutions (also known as repeated solutions)
		- For example, the quadratic equation $x^{2}-10x+25=0$ can be factored into $\left(x-5\right)\left(x-5\right)=0$. It has a repeated solution of $x=5 $.
	+ two non-real complex solutions (contain an imaginary number)
		- Recall that you can solve equations that may have complex solutions using the quadratic formula: $x=\frac{-b\pm \sqrt{b^{2}-4ac}}{2a}$

**Objective 2:** In this section, you apply the Fundamental Theorem of Algebra given the degree of the polynomial to find the number of roots.

*Mathematical Practice Standard: Make sense of problems and persevere in solving them.*

**Big Ideas:**

* The *Fundamental Theorem of Algebra* also gives information about the roots of a polynomial which helps us understand the graph’s properties and shape.
	+ The *degree* of a polynomial directly corresponds to the number of roots the graph will have.
	+ For example, a quadratic, which has a *degree* of two, will have two roots.
* Recall that roots are where a graph crosses the *x*-axis which has an *x* value that is also the solution to the polynomial.
	+ Roots can be real, rational, and complex.

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| **Example** | **Graph** |
| Polynomial: $2x-4=0 $* The polynomial has a degree of one and will therefore have one root.
* The graph shows that there is one root at (2,0).
 |  |

* Recall that real roots will show as an *x*-intercept on a graph, but non-real roots will not show on the graph.
* Recall that a root that is a solution more than once is called a root with multiplicity.

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| **Example:** Apply the Fundamental Theorem of Algebra to find the number of roots for the polynomial. $$2x=-x^{2}-3$$ |
| **Step 1:** Write the polynomial in standard form $ax^{2}+bx+c=0$. | $$2x=-x^{2}-3$$Rearrange by adding $x^{2}$ and 3 to both sides and setting the equation equal to zero.$$x^{2}+2x+3=0$$ |
| **Step 2:** Apply the theorem. | $x^{2}+2x+3=0$ It is a polynomial with a degree of 2, meaning there will be 2 roots. |
| **Step 3:** Graph the polynomial. | Notice that, when graphed, the polynomial does not have any *x*-intercepts. |
| **Step 4:** State the answer.  | The polynomial has two roots that must be complex roots because the graph does not show any *x*-intercepts. |

* The *Fundamental Theorem of Algebra* applies to polynomials with a degree greater than 1.

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| **Polynomial** | **Roots** |
| $$3x^{5}$$ | five roots |
| $$2x^{2}+x+8=-4x^{3}$$ | three roots |
| $$x^{7}+5x^{2}-2x+7=0$$ | seven roots |

**Practice Questions and Answers**

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|  | Question | Answer |
| P 1 | According to the Fundamental Theorem of Algebra, $4x ^{2}+ 11x - 20 = 0$ has two solutions. How many of those solutions are integers? | 1 |
| P 2 | Apply the Fundamental Theorem of Algebra to find the number of roots for the polynomial $g (x) = 7x ^{6}+ 2x – 5$. | 6 |
| P 3 | Apply the Fundamental Theorem of Algebra to find the number of imaginary roots for the polynomial $f (x) = 4x ^{2}- 14 +x ^{8}$, if you know its graph has two *x*-intercepts. | 6 |
| P 4 | What are the roots of $p (x) = x ^{3}+ 4x^{2} - 7x - 10$. Provide the exact answers. Enter the answers in ascending order from the smallest to the largest number. | -5; -1; 2 |
| P 5 | What are the roots of $g (x) = 4x ^{2}+ 256$? Provide the exact answers. Enter the smaller number first. | -$8i; 8i$ |

**Quick Check Questions and Answers**

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|  | Question | Answer |
| Q 1 | Show that the Fundamental Theorem of Algebra is true for the quadratic polynomial $x ^{2}- 16x + 39 = 0$ through solving by factoring. Which of the following statements accurately describes the solution set?  | There are two rational solutions. |
| Q 2 | Without solving, apply the Fundamental Theorem of Algebra to determine how many roots $y = 8x ^{5}- 2x^{4} + 6$ will have. | Five roots |
| Q 3 | Apply the Fundamental Theorem of Algebra to determine how many imaginary roots $g (x) = 12x - 3x ^{2}+ 13x^{3} - 9 $will have, if you know it has one *x*-intercept. | Two imaginary roots |
| Q 4 | Solve the polynomial equation $p (x) = x ^{3}+ 2x ^{2}- 5x - 6$ . Which of the following is a factor? | $$x+1$$ |
| Q 5 | What are the roots of $f (x) = 6x ^{2}+ 216$ ? | $$\pm 6i$$ |